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Applications of Hamiltonian Theory to Plasma Models

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Applications of Hamiltonian Theory to Plasma Models

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Applications of Hamiltonian Theory to Plasma Models

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Three applications of Hamiltonian Methods in Plasma Physics are presented.

The first application is the development of a new, five-field, Hamiltonian gyrofluid model. It is comprised by evolution equations for the ion density, pressure and parallel temperature and electron density and pressure.

It contains curvature and compressibility effects. The model is shown to satisfy a conserved energy and a Lie-Poisson bracket for it is given. Casimir invariants are calculated and through them, the normal fields of the system are recovered. Later, the model is linearized and shown to possess modes that are identified with the slab ITG, toroidal ITG and KBM modes. Both an electrostatic and an electromagnetic study are performed. Growth rates and critical parameters for instability are computed and compared to their fluid and kinetic counterparts. The accuracy of the model is shown to be between the fluid and the kinetic results, as was expected. Dissipation is added to

the ideal system via the use of non-local terms that mimic Landau damping. The modes of the system are shown to undergo Kreĭn bifurcations and their behavior once dissipation is turned on, strongly suggests that they are negative energy modes. A connection between the marginal stability condition of the ITG mode at high $k_\perp$ and the (missing) equation of perpendicular pressure is conjectured opening an interesting possibility for future research.

The second application is a method for the derivation of reduced fluid models through the use of an action principle. The importance of the method lies in the fact that since all approximations are made directly at the level of the action, the models that result from the action minimization are guaranteed to retain the Hamiltonian character of their parent-model. The two-fluid action is given in Lagrangian variables and the two-fluid equations of motion are recovered by it's minimization. The Eulerian (field) equations of motion are retrieved through the Lagrange-to-Euler (L-E) map. New, single-fluid variables are defined but instead of being implemented at the level of the equations of motion, they are implemented directly in the action. The action is subjected to approximations. Different approximations lead to different models with the models of Lüst, Extended MHD, Hall MHD and electron MHD being retrieved. The passing from Lagrangian to Eulerian variables in the single-fluid description requires a non-trivial modification of the E-L map. A note about the importance of quasineutrality in single-fluid models and its ramifications in the Lagrangian framework is given. Several invariants of the models are calculated via Noethers' Theorem.

The third application concerns the imposition of constraints in Hamiltonian systems. Two worked examples of the method of Dirac are presented. The first one is on an electrostatic model which has the Hasegawa-Mima and RMHD as distinct limits. The constraint that leads to the Hasegawa-Mima is investigated. The calculations are demonstrated in detail and the reduced system is produced. A brief discussion of the dispersion relation of the reduced system concludes the first example. The second example is the imposition of quasineutrality and divergence-free current on the bracket of the two-fluid model. The various steps of the method are displayed and the example is completed with the verification that the new bracket satisfies the constraints. The possibility of performing the same calculation with single-fluid variables remains open for future research.

Table of Contents

List of Figures

Μέγα βιβλίον, μέγα κακόν.

Callimachus, 4th century BC

Chapter 1

Introduction

1.1 The need for Hamiltonian models in Plasma Physics

The behavior of plasmas in tremendously different contexts, from astrophysical environments to magnetically confined, thermonuclear plasmas inside tokamaks, can be modeled by systems of partial differential equations that yield the time evolution of key physical variables such as ion and electron density, momentum, pressure, heat flux etc. One usually derives such dynamical equations by taking moments of a distribution function. The equations thus derived are fairly general since they contain the physical description of phenomena that occur over vastly different length and time scales. As a consequence, these exact moment equations are also intractable, both from an analytical and a computational standpoint. To reduce the complexity, it is appropriate that the exact moment equations get subsequently manipulated, according to the particular phenomenon one wishes to model or the specific context that the plasma in question is in, in order to filter out irrelevant time and length scales. This phenomenological process commonly takes the form of small parameter expansions and assumptions about the geometry of the system under consideration. Unfortunately, there is no prescription for this procedure and one has only his or her intuition to rely on. As a result, the sys-

tems of equations produced by such ad hoc procedures often comes with a host of shortcomings. A very serious one is the loss of the Hamiltonian character [63, 130]: The parent model, that is the system of charged particles interacting with an electromagnetic field, is Hamiltonian and as a consequence, it is desirable that any reduced description of it should retain this property. The issue is not just a harmless question of mathematical formalism: The process of reduction might have introduced unwanted dissipation and as a result, the system might violate energy conservation at the ideal limit. By ideal limit, we refer to what remains from the system once all dissipative and source terms such as collisions, Landau damping, anomalous transport and boundary terms have been discarded. To the contrary, a Hamiltonian system is guaranteed to conserve energy for closed boundary conditions and the Hamiltonian formulation is useful for investigating the local properties of the dynamics that are independent of the drive.

Nonetheless, energy conservation is not the sole reason one might have to pursue the discovery of the Hamiltonian formulation of a system. Casting a system into its Hamiltonian form [82, 83] confers several practical advantages. One of the most important is the existence of families of invariants, called Casimir invariants, which are found in noncanonical Hamiltonian systems due to the degeneracy of the cosymplectic matrix. We hope that the discussion in the subsequent sections will make statements like the previous one explicit and convince the reader for the importance of Casimir invariants. For now, suffice

it to say that the functional that results from the addition of the Casimirs to the Hamiltonian has non-trivial equilibrium states as stationary points. In the absence of a Poisson bracket, by contrast, the existence of non-trivial equilibrium states is not guaranteed. For example, Ref. [135] presents an example of a seemingly reasonable fluid model that lacks physical equilibria with closed streamlines because the equilibrium equations imply that some fields are multiple-valued on closed streamlines. We can also take advantage of the Hamiltonian formulation to construct "energy principles" for the investigation of the stability of such non-trivial equilibrium states by examining the second variation of the aforementioned functional. The interested reader can find the description and applications of the so-called Energy-Casimir method in [4, 93, 131]. Another advantage is that imposing constraints on a system is straightforward in the Hamiltonian formalism [16]. The last chapter of this dissertation deals with this topic. Lastly, the Hamiltonian formalism can be used to facilitate the calculation of the statistical average of the zonal flow growth rate [66].

Chapter 2

Review of Canonical and Noncanonical Hamiltonian systems

2.1 Canonical Hamiltonian systems

We will start by giving a brief reminder of the basic ideas of canonical Hamiltonian systems. This review is by no means meant to be complete. For a more exhaustive treatment, the reader is referred to the textbooks [57, 30, 50]. Lets start with a dynamical system that is described by generalized coordinates $q_i, i = 1, \ldots, n$ defined on the configuration space $\mathbf{M^n}$. To each of these we can associate a generalized momentum p_i. If we know the Lagrangian $L(q, \dot{q}, t)$ of the system, then these generalized momenta can be found as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \,. \tag{2.1}$$

The Euler-Lagrange equations are then written as

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q^i} \,, \tag{2.2}$$

$$\frac{dq^i}{dt} = \dot{q}^i \,. \tag{2.3}$$

After we perform a change of variables in the Lagrangian $L(q, \dot{q}, t)$ and express it in terms of the conjugate variables (q^i,p_i) instead, we perform a Legendre transform to find the Hamiltonian of the system: $H(q, p, t) =$

$p_i q^i - L(q,p,t)$. In mathematical language, we say that we pass from the configuration manifold $\mathbf{M^n}$ to the cotangent bundle $\mathbf{T^*M}$. This is called the *phase space* of the system. In order to be able to perform the Legendre transform, we must be careful that the equation (2.1) is invertible. This translates into a condition for the second derivative of L, namely that $\frac{d^2L}{d\dot{q}^2} \neq 0$. If the Lagrangian in question depends on more than one generalised coordinates, the generalisation of the invertibility condition is that the Hessian matrix, $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ must be non-singular. In terms of this new function, the above dynamical equations can be written as:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}\,, \tag{2.4}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}\,. \tag{2.5}$$

In this form they are known as Hamilton's canonical equations. We remark here that whereas the Euler-Lagrange equations are second order and define curves in $\mathbf{M^n}$, Hamilton's equations are first order and define curves in $\mathbf{T^*M}$. This means that in phase space trajectories are separated.

A mathematical object of great importance in Hamiltonian mechanics is the Poisson bracket $\{\cdot,\cdot\}$. It is a map that takes two smooth, real valued functions of phase space (we insert them into one of the two slots) and produces a new smooth, real valued, phase space function. If the function that goes into the right slot is the Hamiltonian, then the Poisson bracket gives the time evolution, under the dynamics, of the function that we insert in the left slot. The mathematical formulation of this statement is given in (2.6) which also

serves as a definition of the Poisson bracket:

$$\dot{f} = \{f, H\} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}\,. \tag{2.6}$$

In terms of the Poisson bracket Hamilton's equations take the following form (suppressing the subscripts)

$$\dot{p} = \{p, H\}\,, \quad \dot{q} = \{q, H\}\,. \tag{2.7}$$

The properties of the Poisson bracket are:

- bilinearity: $\{\lambda f, g + h\} = \lambda\left(\{f, g\} + \{f, h\}\right)$.
- antisymmetry: $\{f, g\} = -\{g, f\}$.
- Leibnitz rule: $\{fg, h\} = \{f, h\}g + f\{g, h\}$.
- Jacobi: $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$.

The next step in the development of the Hamiltonian formalism is to define the symplectic 2-form

$$\omega = dp_i \wedge dq^i\,. \tag{2.8}$$

At first, this definition might seem a little arbitrary. The motivation behind it is the following: We need a 2-form that when we contract it with the Hamiltonian vector field[1] $\mathbf{X}_H = \dot{q}^i\frac{\partial}{\partial q^i} + \dot{p}_i\frac{\partial}{\partial p_i}$ gives us the differential of the

[1]By the subscript of a vector field, we denote the function on which it acts. However, in this definition, by the subscript H we mean the Hamiltonian vector field.

Hamiltonian, dH. In other words, we are looking for ω so that $\mathbf{i}_{X_H}\omega = dH$, which is the geometric form of Hamilton's canonical equations. It turns out that this 2-form is (2.8). Two important properties of this 2-form are:

- $d\omega = 0\,,$
- $\mathbf{i}_X\omega = 0 \quad$ iff $\; X \;$ is a null vector field$\,.$

ω is a bilinear and antisymmetric 2-form that sends pairs of vector fields to functions. It can locally be described by a $2n \times 2n$ matrix of the following form: $J_c = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$, where 0_n and I_n are the $n \times n$ zero and identity matrices respectively.

One of the big advantages of the Hamiltonian formalism is that it treats both coordinates and momenta on equal footing. Therefore, we can define new variables $z_i, i = 1, \ldots, 2n$ with

$$z^i = q^i\,, i \in (1, \ldots, n)\,, \tag{2.9}$$

$$z^i = p^{i-n}\,, i \in (n+1, \ldots, 2n)\,. \tag{2.10}$$

Using these unified coordinates we can relate the Poisson bracket and the ω 2-form by:

$$\{f, g\} = \omega(X_f, X_g) = \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial g}{\partial z^j}\,. \tag{2.11}$$

with X_f and X_g being the Hamiltonian vector fields of f and g defined by the following relation:

$$\mathbf{i}_{X_f}\omega = df\,. \tag{2.12}$$

2.2 Noncanonical Hamiltonian Systems

Many times we are confronted with dynamical systems whose time evolution is not described by an equation such as Eq.(2.6) and we have to decide whether they are Hamiltonian or not. Before we answer the question of how we can tell if a dynamical system is Hamiltonian, it is instructive to show how we can start from a canonical Hamiltonian system and perform a transformation of variables under which, the system is no longer characterized by the evolution equation Eq.(2.6). Lets start with a canonical Hamiltonian system and imagine a time-independent change of variables to our canonical coordinates

$$\bar{z}^i = \bar{z}^i(z)\,. \tag{2.13}$$

The Hamiltonian undergoes the same transformation $H(z) = \bar{H}(\bar{z})$. Taking the time derivative of Eq. (2.13), using Eq. (2.11) gives:

$$\dot{\bar{z}}^l = \frac{\partial \bar{z}^l}{\partial z^i}\dot{z}^i = \frac{\partial \bar{z}^l}{\partial z^i} J_c^{ij} \frac{\partial H}{\partial z^j} = \left[\frac{\partial \bar{z}^l}{\partial z^i} J_c^{ij} \frac{\partial \bar{z}^m}{\partial z^j}\right] \frac{\partial \bar{H}}{\partial \bar{z}^m}\,. \tag{2.14}$$

We see that the evolution is no longer given by Eq.(2.6) However, if we define a new matrix J to be

$$J^{lm} = \frac{\partial \bar{z}^l}{\partial z^i} J_c^{ij} \frac{\partial \bar{z}^m}{\partial z^j}\,, \tag{2.15}$$

Hamilton's equations take the suggestive form:

$$\dot{\bar{z}}^l = J^{lm}(\bar{z}) \frac{\partial \bar{H}}{\partial \bar{z}^m} = \{\bar{z}^l, \bar{H}\}\,, \tag{2.16}$$

with a Poisson bracket defined as:

$$\{f, g\} = \frac{\partial f}{\partial \bar{z}^l} J^{lm} \frac{\partial g}{\partial \bar{z}^m}\,. \tag{2.17}$$

J is no longer in the canonical form and, in general, depends on z^i. However, because the bracket still needs to satisfy bilinearity, antisymmetry and the Jacobi identity, the new, *noncanonical, co-symplectic* matrix J needs to have the following properties:

- $$J^{ij} = -J^{ji}\,, \tag{2.18}$$

- $$J^{il}\frac{\partial J^{jk}}{\partial z^l} + J^{jl}\frac{\partial J^{ki}}{\partial z^l} + J^{kl}\frac{\partial J^{ij}}{\partial z^l} = 0\,. \tag{2.19}$$

In fact, it is the existence of a co-symplectic matrix J, satisfying properties (2.18)-(2.19), in terms of which we can write the time evolution of a dynamical system as Eq.(2.16) that guarantees that this system is a Hamiltonian system. It is in the properties (2.18)-(2.19) that lies the Hamiltonian character of a system.

One might wonder why would anyone perform such a coordinate tranformation and get himself in all this trouble in the first place. The answer to this question is that usually, it is nature that has already done this. In most cases of models describing ideal, continuous media the 'physical' variables are noncanonical. However, due to a Theorem of Darboux, if we have a J that satisfies antisymmetry and the Jacobi identity and moreover is non-singular, there always exists a transformation that can take us back to J_c. If, on the other hand, $detJ = 0$ with J having a rank $2k < 2n$ then, according to a

generalization of Darboux's Theorem attributed to Lie [74], we can find a transformation that takes J to:

$$J_c = \begin{bmatrix} 0_k & I_k & 0 \\ -I_k & 0_k & 0 \\ 0 & 0 & 0_{2n-2k} \end{bmatrix}.$$

By the form of the above matrix it is clear that the system looks like a k degree of freedom, canonical Hamiltonian system with $n-k$ extraneous coordinates. Because of this degeneracy, there exist geometrical constants of motion that are built-in the phase space. This makes them invariant under *any* choice of Hamiltonian. They are the so-called *Casimir Invariants*. Because they are invariants for any Hamiltonian, their gradients span the null space of J as we can see by using the definition of a non-canonical Poisson bracket:

$$\{F, C\} = J^{ij}\frac{\partial C^{\alpha}}{\partial z^j} = 0\,. \tag{2.20}$$

What we have discussed so far pertains to systems with finite degrees of freedom. When we attempt to model continuous systems such as magnetofluids, we are going to have to formulate them in terms of Eulerian field variables such as density, velocity, pressure etc. Then we necessarily need to work with infinite-dimensional systems. The infinite-dimensional analogue of Eq.(2.17) is:

$$\{F, G\} = \int_D \frac{\delta F}{\delta \psi^i}\mathcal{J}^{ij}\frac{\delta G}{\delta \psi^j}d\mu\,, \tag{2.21}$$

with F and G now being functionals, $\psi^i(\mu, t)$'s being field variables, and μ being Eulerian observational variables. Gradients have been replaced by functional derivatives, whereas $\mathcal{J}$ is now an operator that needs to satisfy relations

(2.18)–(2.19) but for functionals. Brackets of the form Eq.(2.21) are called Lie-Poisson brackets, under the condition that J is linear in the ψ's. More information about noncanonical Hamiltonian systems and their geometric structure can be found in the review [83] on which this subsection is based.

Chapter 3

A Hamiltonian five-field Model for ITG

In the following chapter, we will present a Hamiltonian, five-field, gyrofluid model and use it to study the ITG instability. The results of this study can be found in [60] which constitutes the backbone for this chapter. The chapter is organized as follows: In Section 3.1 we explain the significance of drift wave turbulence for the study of fusion plasmas and outline some arguments for why one would want to study them using gyrofluid models. In Section 3.2 we present some, mainly qualitative, features of drift waves in general and the ITG mode in particular. In Section 3.3 we give the normalizations of our variables, we present the ideal limit of the dynamical model and give some connections with previous work in the field. In Section 3.4 we give the Hamiltonian formulation of the model equations by providing a conserved energy that serves as the Hamiltonian and a Lie-Poisson bracket that satisfies the Jacobi identity. In Section 3.5 we calculate the Casimir invariants of our system and from them, in Section 3.6 we construct five "normal fields" which are field variables in which the dynamical equations and the bracket take a very simple form. Lastly, in Section 3.7 we perform a local, linear study of the model with particular emphasis on the study of the ITG and KBM modes. We present stability criteria for both the ideal model and a model with linear

dissipation terms representing the effects of parallel Landau damping and the drift resonance. We investigate several well-known stabilizing factors of the instability to show qualitative agreement with kinetic models.

3.1 Motivation

We believe that in Sec. 1.1 we provided ample justification for why we desire the models we build to retain their Hamiltonian character. In this section, we wish to explain the motivation behind studying drift waves and why would one wish to do so using gyrofluid models in particular.

In all plasma experiments, care must be taken so that the extremely hot, fusion-grade plasma stays confined at the core and doesn't hit the walls of the machine. This results in the formation of a region near the edge of the plasma where density and temperature profiles drop abruptly, forming sharp gradients. This region is known as the pedestal. Because of the intense pedestal gradients, instabilities are excited that tap the free energy of the configuration seeking to straighten the profiles and diminish the gradients. These modes are collectively called drift waves. For relevant tokamak parameters (low β and low collisionality), the dominant mode is the Ion Temperature Gradient (ITG) mode which gets destabilized by the presence of an ion temperature gradient. Moreover, it is widely believed that in the parameter range that most tokamak experiments operate, the ITG mode is always unstable. A mode however, cannot remain unstable forever. Inevitably, at some point it reaches saturation. From this instability-saturation cycle, a new stable equilibrium emerges which

we call marginal stability. It is, roughly speaking, the point where the mode is *about* to go unstable. As a result of marginality, the temperature at the core of the plasma is *multiplicatively* related to the temperature at the pedestal [65, 37]. This makes the study of ITG and drift wave modes all the more important since it is their marginal stability that sets the temperature profile of the tokamak and determines the temperature at the core, which is what we ultimately care about if we want to achieve fusion.

An additional reason for the interest of the plasma community in drift waves is that they are believed to be responsible for the anomalous transport in tokamaks. Tokamak experiments always record a level of transport, and the concomitant heat and particle losses, that is much higher than what one would expect from collisions or neoclassical transport. This has a direct effect in the realization of fusion because it lowers the confinement time in Lawson's criterion.

A simple analysis of drift waves shows that their spatial scale is of the size of the ion gyroradius, $k_\perp \rho_i \sim 1$, with $\rho_i = \dfrac{u_{ti}}{\Omega_i}$ and $u_{ti} = \sqrt{\dfrac{T_i}{m_i}}$ being the ion thermal velocity. Also, they are found to grow at the rate of the diamagnetic frequency $\omega_* = \dfrac{k_y \rho_i u_{ti}}{L_n}$ with $L_n = -\dfrac{1}{n_o}\dfrac{dn_i}{dr}$ being the ion density scale length. If we take these two as a starting point, we can make a mixing length estimate for the diffusivity by assuming that particles follow a random walk with step equal to ρ_i and frequency of steps equal to ω_*:

$$D \sim \frac{(\Delta x)^2}{\Delta t} \sim \frac{\gamma}{k_\perp^2} \sim \frac{1}{k_\perp \rho_i}\frac{\rho_i^2 u_{ti}}{L_n} \sim \frac{\rho_i^2 u_{ti}}{L_n}\,, \tag{3.1}$$

which is the so-called "gyro-Bohm" diffusivity, D_{gB}. One of the main problems of this scaling is that the transport it predicts is usually much lower than the one observed in actual experiments. For a more comprehensive list of problems with the gyro-Bohm scaling of diffusivity, the interested reader is referred to the introduction of [8] where a very thorough historical review (up to 1995) of the field of microinstabilities research is also given.

The lesson that can be learned by the apparent failure of mixing length arguments to predict the transport level is that turbulence is inherently nonlinear and three dimensional [8]. Therefore, attempts at calculating heat and particle diffusivities analytically are quite challenging and severely limited. As a consequence, our only option to comprehend these phenomena in a quantitatively accurate way is to do simulations in realistic geometries with models that incorporate all relevant effects. An obvious class of suitable models would be kinetic in nature. Since no truncation of the moment hierarchy is taking place, we don't have to worry about excluding important phase space effects. However, because curvature plays a tremendously important role in the growth rates of drift modes, in order to draw accurate physical conclusions, we need to carry out the simulations in realistic tokamak geometry which adds to the complexity of kinetic simulations and makes them remarkably computationally expensive. Consequently, if we want an easy to use, agile tool to gain physical insight for microturbulence in tokamak plasmas, we need to turn to fluid models.

Fluid models are derived by taking moments of the Vlasov equation.

After such an infinite number of moments is generated, a truncation at some point of this moment hierarchy is performed. Because the equation for the n-th moment always depends on the $n+1$ moment, a closure scheme must necessarily be employed. Owing to this inevitable closure problem, fluid models fail to include many phase space effects such as Landau damping or the effect of trapped particles. Fortunately, many attempts have been made to successfully introduce terms in fluid models that mimic kinetic effects [34, 113, 25, 136].

By deciding upon a closure scheme for the truncation of the moment hierarchy and further manipulation of the resulting equations in order to omit irrelevant time and length scales we arrive at reduced fluid models. Such models constitute versatile tools for the study of multi-scale phenomena including, in particular, the interaction of turbulence with magnetohydrodynamic perturbations exhibiting meso-scale structures.[133] Examples include magnetic islands,[56, 47] edge localized modes,[140, 139] resonant magnetic perturbations,[80, 18] as well as fishbone [101] and Alfvén modes.[115, 128]

Among the several classes of fluid models, of particular importance are the ones that retain the effects of finite ion temperature, principally for describing instabilities with growth rates comparable to the ion diamagnetic frequency or modes with perpendicular wavelengths of the order of the ion Larmor radius. Whereas "cold ion" models have been shown to possess noncanonical Hamiltonian formulations,[89, 123] the task of formulating such "hot-ion" models that satisfy the Hamiltonian property has proven difficult. For example, efforts to identify the Hamiltonian structure of the four-field model of Ref. [41]

were unsuccessful, even though it conserves energy.[42] The main difficulty with such models lies in the nonlocality of the ion dynamics caused by Larmor gyration. One way to approximate nonlocal terms is by a Taylor-series, using $k_\perp \rho_i$ as a small parameter. An example of such a so-called FLR model was given in Ref. [40], where a Hamiltonian four-field model is constructed, using the "gyromap" technique to introduce finite ion temperature into the cold ion limit of Ref. [41]. Unfortunately, we are unaware of any numerical implementation of this model, possibly because it requires high-order derivatives and, consequently, additional boundary conditions.

An alternative approach for constructing fluid models with a finite ion temperature is to truncate the moment hierarchy of the gyrokinetic equation instead of the Vlasov equation [25, 11, 113, 112, 111]. This leads to the use of nonlocal averaging operators that account for the full range of perpendicular wavelengths. The resulting models are called gyrofluid models. Surprisingly, gyrofluid models are more readily amenable to Hamiltonian formulations than FLR models. Examples of Hamiltonian electromagnetic gyrofluid models are given in Ref. [134] for an incompressible (three fields) and Ref. [132] for a compressible (four fields) model. The four-field gyrofluid model advances the first two moments of the distribution function for each species, or the ion and electron densities and parallel momenta. Zacharias *et al.* have shown that simulations of magnetic reconnection using this model are in good agreement with gyrokinetic simulations,[142] and Comisso *et al.* have used it to bring to light the acceleration of magnetic reconnection by nonlocal gyrofluid

effects.[19] Grasso *et al.*, by contrast, have used it to examine the stabilizing effects of ion diamagnetic drifts on the growth and saturation of tearing modes in inhomogeneous plasma.[32]

Given the wide availability of several high-quality gyrokinetic (GK) codes that have been verified and validated in a broad array of contexts, it is appropriate to reflect on the value of gyrofluid (GF) models. Due to their nature as truncated moment expansions of the GK model, GF models such as the one presented here cannot aspire to compete with the latter in any but three domains: speed, ease of use, and by virtue of the first two, ability to generate physical insight. The success of the TGLF code [117, 49, 64] demonstrates that there is a strong demand for an agile quasilinear GF code to understand and interpret experimental observations of turbulent transport. The motivation for the development of the Hamiltonian GF model that we will be presenting over the next sections is similar but different: it is to provide an equally agile tool to investigate *multi-scale nonlinear* problems such as those listed above. In this context, the linear accuracy of the model is of secondary importance compared to assuring the proper conservation laws and providing a qualitatively correct picture of the nonlinear energy transfers. It is worth noting, in this context, that a Poisson bracket for a gyrokinetic model, demonstrating its Hamiltonian nature, has only recently been constructed [12] using the newly developed technique of gauge-free lifting.[86]

3.2 Review of the ITG mode

3.2.1 General Features of Drift Waves

Before we give a review of the ITG mode, it is instructive to provide a quick sketch of drift waves in general. The discussion will be based on Fig. 3.1. There, we depict a slice of the outer midplane of a tokamak with a density gradient. Drawing an arbitrary vertical line, we divide the plane in two areas: a "more" dense and a "less" dense ones. We imagine an ion density perturbation which is slow enough so that the electrons are always in equilibrium. The ion excursion in the upper part of the picture creates a high-density patch inside a low-density ion population. The electrons that move freely along the field lines, rush in to restore balance creating a positive potential at that spot. The opposite situation takes place when the perturbation creates a low-density patch in the high-density region. Electrons leave the region, leaving behind them an area with negative potential. The result is that an electric field is established. This E-field, in combination with the imposed, toroidal magnetic field, sets up an $E \times B$ drift on the particles whose direction is such as to restore the equilibrium, pushing the high density patch back to the high-density region and the low-density patch to the low-density region. Instability occurs when the electron response is, for some reason, out of phase with the initial ion perturbation.

Now, we will attempt to put the above description in mathematical terms and do a back-of-the-envelope calculation that will reveal the basic scalings of drift-wave dynamics. Because of quasineutrality, we expect the ion

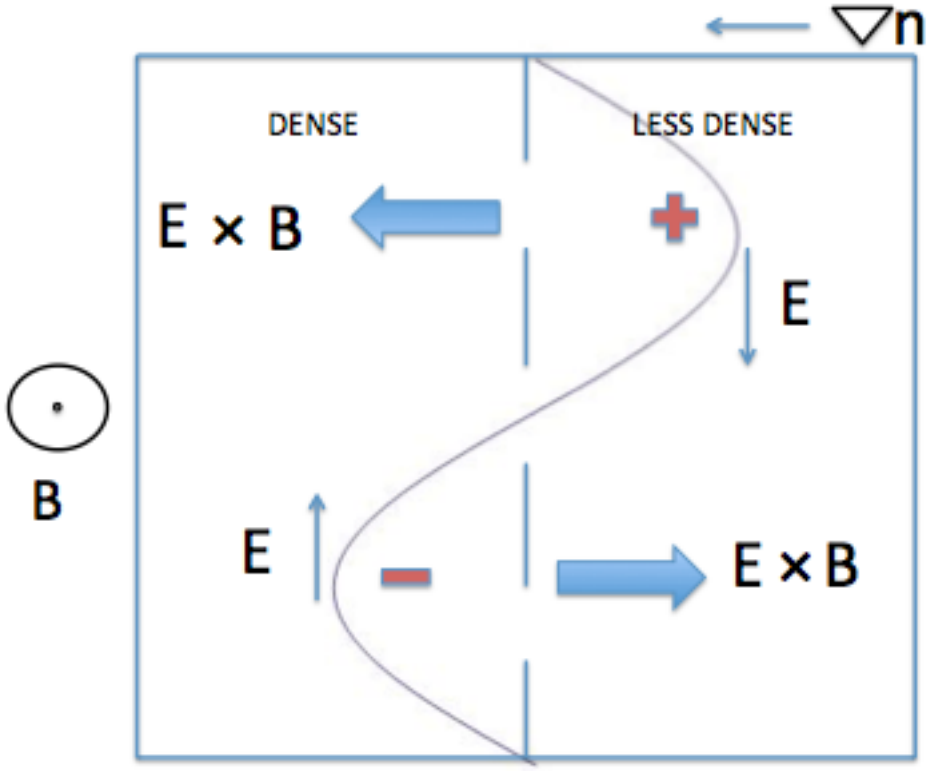


Figure 3.1: Mechanism of generation of drift waves

background density plus the density perturbation to always equal the electron (equilibrated) density:

$$n = n_o e^{-\frac{e\phi}{T_e}} = n_o + \delta n_i \,, \tag{3.2}$$

where we have set the Boltzmann constant, k_B, equal to one for convenience. This statement implies that the density perturbations we are considering have a wavelength larger than the Debye length, otherwise, we would have to allow for charge imbalance. We define the ion displacement vector ξ and, after we express the ion density in terms of it as $n_i = n_o + \nabla n \cdot \xi$ and Taylor expand the electron density, we solve for the potential:

$$n_o + \nabla n \cdot \xi = n_o \left(1 - \frac{e\phi}{T_e}\right) \,, \tag{3.3}$$

$$\phi = -\frac{T_e}{n_o e} \nabla n \cdot \xi \,. \tag{3.4}$$

It is straightforward to calculate the electric field:

$$E = -\nabla\phi = \frac{k_y T_e}{e} \frac{\nabla n}{n_o} \xi \,. \tag{3.5}$$

where we have used the fact that spatial variation of perturbed quantities is of the order of wavelenght, i.e, $\nabla(\delta f) \sim k_y \, \delta f$. Now, we are in a position to compute the magnitude of the $E \times B$ drift:

$$V_{E\times B} = c\frac{E \times B}{B^2} = \frac{c k_y T_e \nabla n}{eBn_o} \xi = \omega_* \xi \,, \tag{3.6}$$

with $\omega_* = \dfrac{c k_y T_e}{eB} \dfrac{\nabla n}{n_o}$. With this simple calculation, we arrived at the diamagnetic frequency ω_* as the basic time scale for drift-waves.

3.2.2 An Intuitive Picture of the ITG Mode

Here we will give a simple picture of the ITG mode based on the description in [20]. Our analysis will be purely qualitative and will focus on the different origins of the slab and the toroidal versions of this instability. The five-field model that will follow, which constitutes the main subject of the chapter, will ultimately make the description quantitative.

In the previous section we gave a qualitative picture of drift waves in general. There, the only gradient was a density gradient. The ITG mode, as the name suggests, is caused by a temperature gradient. The slab version of the mode can be summarized in Fig. 3.2. Again, as in Fig. 3.1, we draw an arbitrary line separating the plasma into a "hot" and a "cold" region because of the temperature gradient. The ITG mode is a negative compressibility mode. This means that somewhere in the plasma, there is a compression of the ions and the dynamics work out in such a way that instead of opposing the compression and restoring equilibrium, they actually enhance it. To see how such a thing might be accomplished, consider a spot where there is an ion density build up. Because the $E \times B$ drift velocity is incompressible (only when the magnetic field is uniform, as it is in the slab case) this build up can only be caused by motion along the field lines. The electrons, being adiabatic, respond by moving along the field lines to maintain quasineutrality, setting up a potential. This potential, establishes an electric field which causes an $E \times B$ drift. This drift is such that it injects cool ions into the "hot" (compressed) region. The result is that the ion pressure is lowered locally, drawing ions

along the field lines by generating a $u_{\parallel}$ to go and cover the low pressure spot. Now, if all the phases work out correctly (which depends on many factors such as the magnitude of the temperature and density scale lengths), since the whole picture develops in time and moves perpendicular to both the magnetic field and the temperature gradient, the ions that move parallel to the field line, prompted by the lowered pressure, end up increasing the initial density perturbation. It will be useful to keep this simplified picture of the dynamics when considering the full dispersion relation that comes from the model. It will give us intuition for whether different factors behave in a stabilizing or destabilizing way.

To explain the toroidal version of the instability we use an argument from [8] and we refer to Fig. 3.3. This instability depends on "good" and "bad" curvature effects. Here we pause to explain what we mean by "good" and "bad" curvature. In a tokamak, the particles follow the field lines and they feel a centrifugal force due to their poloidal velocity. This centrifugal force, is always pointing to the outer region of the tokamak ($F_{CF} = m\omega \times (\omega \times \mathbf{r})$). In the outer midplane of the tokamak, this force will be in the opposite direction of the pressure gradient. Thus, we speak about a "bad" curvature region since F_{CF} wants to push the particles towards the wall of the machine. In the inner midplane, centrifugal force and pressure gradient will be parallel to each other and we speak about a "good" curvature region, where F_{CF} helps keep the particles away from the wall.

Toroidicity and non-uniformity of the magnetic field, induce drifts on

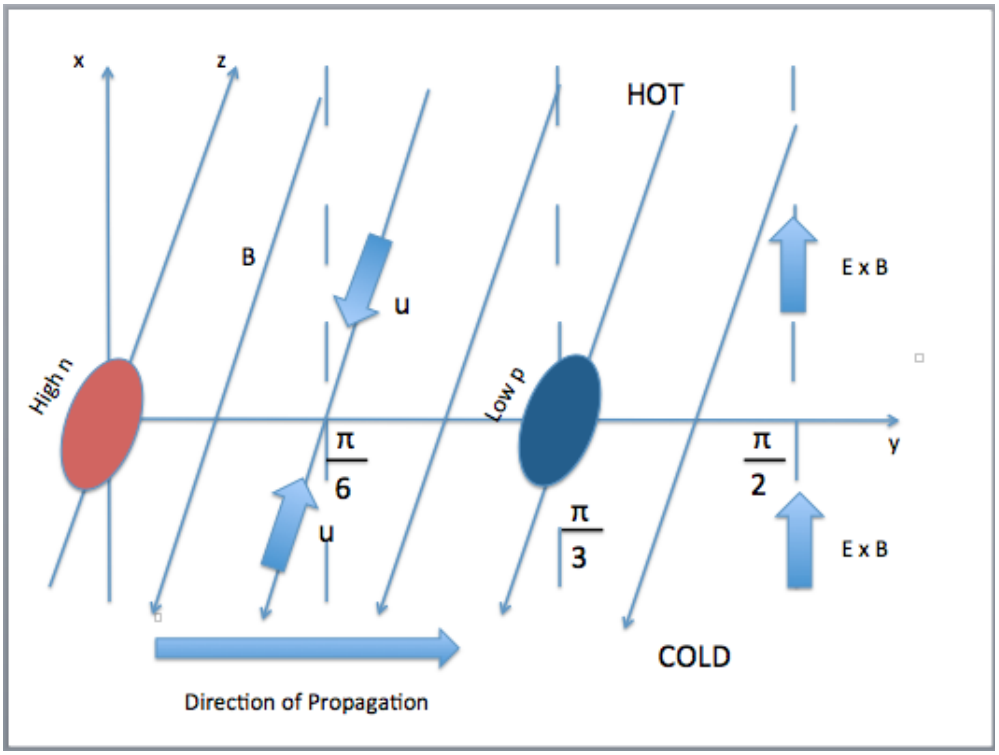


Figure 3.2: Feedback mechanism of slab ITG

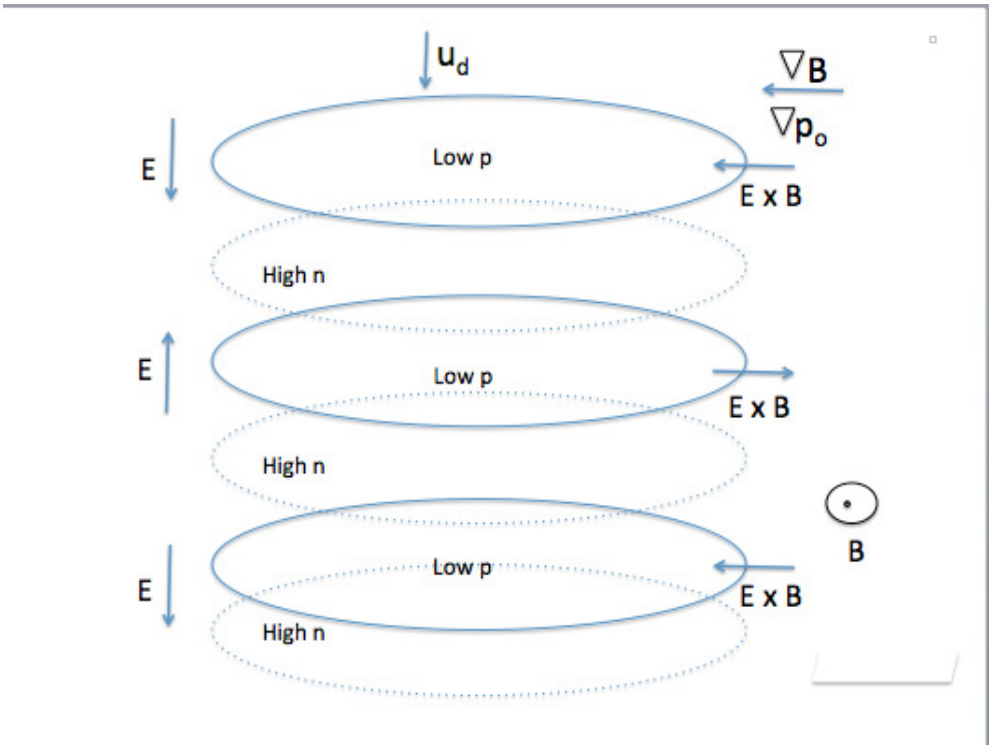


Figure 3.3: Mechanism of toroidal ITG at the outer midplane of the Tokamak.

the particles called *curvature* and ∇B drifts, respectively. Because the two drifts have similar vector forms they can be combined into a single equation:

$$u_d = \frac{u_\parallel^2 + \frac{u_\perp^2}{2}}{\Omega B^2} \mathbf{B} \times \nabla \mathbf{B} \,. \tag{3.7}$$

We will henceforth call this combined drift, the gradient drift. From the form of the above equation we see that the gradient drift u_d depends on the velocity of the particles. Therefore, "hot" particles drift faster than "cold" ones. In the outer midplane of the tokamak, the magnetic field gradient is such that the gradient drift is pointing down. This gradient drift will create an ion density perturbation because "hot" ions gradient-drift faster than cold ones, creating increased density patches under hot patches and lowered density patches under cold patches. As explained in the previous paragraphs, such an ion density perturbation will lead to movement of electrons and to the establishment of a potential and, eventually, an electric field in the direction shown in Fig. 3.3. This electric field will cause an $E \times B$ drift which will bring cool ions into the already cool ion region and hot ions into the already hot ion region, driving the mode unstable. In the inner midplane of the tokamak, the pressure gradient reverses whereas the magnetic field gradient stays the same. In this case, the ensuing $E \times B$ drift will bring hot ions into the cold ion regions and cold ions into the hot ion regions, restoring the equilibrium.

3.3 Ideal Five-Field Model

The model we propose is a Hamiltonian five-field electromagnetic gyrofluid model that is an extension of the model presented in Ref. [132]. The new model, like its predecessor, is a truncation of a more complete one proposed by Snyder and Hammett, which advances six moments for the ions and two moments for the electron dynamics [113]. We note that Scott [112, 111] has shown that achieving energy conservation requires modifying several of the terms in Ref. [113] involving higher order moments. We will likewise show that constructing a *Hamiltonian* model requires modifying the terms involving the higher order moments in our model. The new model extends that in Ref. [132] by the addition of the evolution of the ion temperature. As in the previous model, ion compressibility effects and field curvature are also included, allowing it to describe ITG, KBM, drift waves and tearing modes. To demonstrate the properties of the model, we present a linear, local study of electrostatic slab ITG and toroidal electromagnetic ITG modes. The results of this study have been published in Ref. [60].

We first present the ideal portion of our model by omitting collisional diffusion and wave-particle interaction terms, which will be examined in Sec. 3.7.

We are interested in a model that describes the destabilization of the drift wave excited by the ion temperature gradient. Due to the acoustic nature of the instability, we cannot neglect ion motion along the field lines; therefore, we keep ion compressibility effects. Also, because we want to investigate toroidal plasma with finite β, we include electromagnetic effects.

Lastly, to represent the influence of toroidicity, we allow for magnetic curvature. We consider the evolution of the the magnetic flux ψ, of a magnetic field $\mathbf{B} = \hat{\mathbf{z}} + \nabla\psi \times \hat{\mathbf{z}}$, the ion density n_i, the parallel velocity of the ion *guiding centers* $u_i = \hat{\mathbf{z}} \cdot \mathbf{v_i}$, the electron density n_e and parallel velocity $u_e = \hat{\mathbf{z}} \cdot \mathbf{v_e}$, the electrostatic potential ϕ and the parallel ion temperature $T_\parallel$. We normalize these quantities in the following way:

$$(n_i, n_e, \psi, \phi, u_i, u_e, T_\parallel) = \frac{L_n}{\rho_i}\left(\frac{\hat{n}_i}{n_o}, \frac{\hat{n}_e}{n_o}, \frac{\hat{\psi}}{\rho_i B_o}, \frac{e\hat{\phi}}{\tau T_i}, \frac{\hat{u}_i}{v_{ti}}, \frac{\hat{u}_e}{v_{ti}}, \frac{\hat{T}_\parallel}{T_i}\right), \tag{3.8}$$

where the carets denote the dimensional variables. Here n_o, B_o and T_e are the background density, magnetic field and electron temperature, $\rho_i = v_{ti}/\omega_{ci}$ is the ion Larmor radius, where $v_{ti} = (T_i/m_i)^{\frac{1}{2}}$ is the ion thermal speed, $\omega_{ci} = eB_o/m_i$ is the ion cyclotron frequency, $L_n = n_o/|\nabla n|$ is the density scale-length and $\tau = T_e/T_i$ is the ratio of the species temperatures. We also normalize the independent variables according to:

$$(t, k_\parallel, k_\perp) = \left(\frac{\hat{t}v_{ti}}{L_n}, \hat{k}_\parallel L_n, \hat{k}_\perp \rho_i\right). \tag{3.9}$$

With these normalizations, our evolution equations are as follows. The equations that describe the ideal evolution of ion quantities are

$$\frac{dn_i}{dt} = -\nabla_\parallel u_i - 2u_d \frac{\partial}{\partial y}(n_i + \Phi + T_\parallel), \tag{3.10}$$

$$\frac{d(\Psi + u_i)}{dt} = -\nabla_\parallel T_\parallel - \nabla_\parallel n_i - 4u_d \frac{\partial u_i}{\partial y}, \tag{3.11}$$

$$\frac{dT_\parallel}{dt} = -(\gamma - 1)\nabla_\parallel u_i - 2u_d \frac{\partial}{\partial y}(n_i + \Phi + T_\parallel), \tag{3.12}$$

whereas the equations describing the evolution of electron quantities are

$$\frac{dn_e}{dt} = -\nabla_\parallel u_e + 2u_d \frac{\partial}{\partial y}(n_e - \phi)\,, \tag{3.13}$$

$$\frac{d(\psi - \mu u_e)}{dt} = \frac{1}{\tau}\nabla_\parallel n_e + 2\mu u_d \frac{\partial u_e}{\partial y}\,. \tag{3.14}$$

In Eqs. (3.10)–(3.14), $df/dt = \partial f/\partial t + [\Phi, f]$ and $\nabla_\parallel f = \partial f/\partial z - [\Psi, f]$, with $[\cdot,\cdot]$ denoting the canonical Poisson bracket, so that $[f,g] = \hat{\mathbf{z}} \cdot (\nabla f \times \nabla g)$. Also, γ is the adiabatic index, $u_d = L_n/R$ is the normalized curvature drift velocity, R is the radius of curvature of the magnetic field and $\Phi = \Gamma_o^{1/2}\phi$, $\Psi = \Gamma_o^{1/2}\psi$ are the gyro-averaged ϕ and ψ. The symbol $\Gamma_o^{1/2}$ refers to the gyroaveraging operator introduced in Ref. [25] and is defined by

$$\Gamma_o^{1/2}\xi = \exp\left(\frac{1}{2}\nabla_\perp^2\right) I_o^{1/2}\left(-\nabla_\perp^2\right)\xi\,, \tag{3.15}$$

where I_o is a modified Bessel function of the first kind and the result of Eq.(3.15) should be interpreted in terms of its series expansion. At this point, we note that only the ion guiding centers respond to the gyroaveraged value of the electromagnetic field. Therefore, we are required to use the gyroaveraged value of the electrostatic potential in the $\mathbf{E}\times\mathbf{B}$ drift advecting the ions, whereas electrons are advected only by the local value of their $\mathbf{E}\times\mathbf{B}$ drift since we neglect the electron Larmor radius.

Equations (3.10)–(3.14) are closed by the parallel component of Ampére's law

$$\frac{2}{\tau\beta_e}\nabla_\perp^2\psi = -j = -\Gamma_o^{1/2}u_i + u_e, \tag{3.16}$$

with $j = \hat{\mathbf{z}} \cdot \mathbf{J}$ being the z-component of the current density, and by the quasineutrality condition

$$n_e = \Gamma_o^{1/2} n_i + (\Gamma_o - 1)\,\phi, \tag{3.17}$$

with $\Gamma_o = \left(\Gamma_o^{1/2}\right)^2$. Here, $\Gamma_o^{1/2} n_i$ is the gyrophase-independent part of the real space ion particle density and the $(\Gamma_o - 1)\phi$ term comes from the gyrophase-dependent part of the distribution function. It represents the ion polarization density due to the variation of the electric field around a gyro-orbit. We leave β_e unrestricted so that we can describe both "inertial" ($\beta_e \ll \mu$) and "kinetic" ($\beta_e \gg \mu$) Alfvén waves. Since our only temperature equation involves the parallel temperature, from now on we will drop the subscript from $T_\parallel$.

It is interesting to compare the model presented in equations (3.10)–(3.14) to one obtained from the models of Refs. [113, 112, 111] by discarding all the terms involving high-order moments and associated terms. By "associated" terms, we mean for example that discarding $T_\perp$ requires that one also discard terms involving the gyroaveraging operator J_1, since the latter terms result from the effects on gyroaveraged quantities of the variations in the perpendicular temperature. The link between $T_\perp$ and J_1 is reflected in the fact that for energy conservation, J_1 terms must appear together with $T_\perp$, as noted in Refs. [112, 111]. The omission of the terms containing J_1 means, in effect, that we neglect ∇J_0. Compared to such a truncated model, the Hamiltonian model in Eqs. (3.10)–(3.14) lacks any trapped particle effects (terms proportional to $\nabla_\parallel B$ in Refs. [113, 112, 111]) and has a less accurate treatment of

FLR terms (due to the omission of the J_1 terms). The two models also differ in the coefficients of the various curvature terms. In the continuity equation, for example, the argument of the curvature operator in the truncated version of the model of Refs. [113, 112, 111] is $\Phi + p_\parallel/2$, while that in our model is $\Phi + p_\parallel$. This difference is necessary in order for the five-field model to conserve energy. In fact, we note that the curvature terms in Eqs. (3.10), (3.12) and (3.13) are the same as the ones found in the corresponding equations of the FLR fluid model of Ref. [143], which evolves three ion moments, as we do, and conserves energy. Lastly, we note that the factor of four in front of the curvature term in the momentum equation, Eq. (3.11), does match the corresponding term in Refs. [113, 112, 111] despite the fact that for the four-field model of Ref. [132], satisfying the Jacobi identity required halving this factor. The conclusion of these observations is that constructing Hamiltonian models requires modifying the truncated moment expansions, but that the correct terms are recovered as one increases the order of the model.

3.4 The Hamiltonian Form

The system described in Sec. 3.3 conserves the following energy:

$$H = \frac{1}{2}\int_{\mathcal{D}} d^2x \left(\frac{n_e^2}{\tau} + n_i^2 + \frac{1}{\gamma - 1} T^2 + \mu u_e^2 + u_i^2 + \tfrac{2}{\tau\beta_e}|\nabla\psi|^2 + \Phi n_i - \phi n_e \right) , \tag{3.18}$$

where $\mathcal{D}$ denotes the spatial domain of interest and the boundary conditions are such that surface terms vanish. The successive terms of the functional

of Eq.(3.18) represent, respectively, the electron and (two terms) ion thermal energies (for an explanation about how such terms might appear in the energy integral, the interested reader might find Ref. [43] enlightening), the parallel component of the electron and ion kinetic energies, the magnetic energy and the electrostatic energies of ions and electrons. Taking the energy functional as the Hamiltonian of our 5-field model, we can write the set of equations in a noncanonical[83] Hamiltonian form

$$\frac{\partial \xi^i}{\partial t} = \{\xi^i, H\}, \quad i = 1, \ldots, 5, \tag{3.19}$$

with ξ^i being the field variables and $\{\cdot, \cdot\}$ being a noncanonical Poisson bracket. We employ the dynamical variables n_i, M_i, n_e, M_e, T, where $M_i = \Gamma_o^{1/2}\psi + u_i$ is the canonical ion momentum and $M_e = \psi - \mu u_e$, the electron one. Additionally, we define $\tilde{n}_i = n_i - 2u_d x$, $\tilde{n}_e = n_e - 2u_d x$ and $\tilde{T} = T - 2u_d x$ for convenience.

In these variables, the bracket given by

$$\begin{aligned}\{F,G\} = \int d^3x \Big(& -\tilde{n}_i([F_{n_i},G_{n_i}]+[F_{M_i},G_{M_i}] \\ & +[F_T,G_T]) - M_i([F_{M_i},G_{n_i}]+[F_{n_i},G_{M_i}] \\ & +([F_T,G_{M_i}]+[F_{M_i},G_T])) \\ & -\tilde{T}([F_{n_i},G_T]+[F_T,G_{n_i}]+[F_{M_i},G_{M_i}]) \\ & +\tilde{n}_e([F_{n_e},G_{n_e}]+\mu[F_{M_e},G_{M_e}]) \\ & +M_e([F_{M_e},G_{n_e}]+[F_{n_e},G_{M_e}]) \\ & -(F_{M_i}\partial_z G_{n_i} - G_{M_i}\partial_z F_{n_i}) \\ & -(F_T\partial_z G_{M_i} - G_T\partial_z F_{M_i}) \\ & +(F_{M_e}\partial_z G_{n_e} - G_{M_e}\partial_z F_{n_e})\Big) \end{aligned} \tag{3.20}$$

satisfies the formulation of Eq. (3.19) for the Eqs. (3.10)–(3.14), is bilinear, antisymmetric and satisfies the Jacobi identity. In the above bracket, we have taken $\gamma = 2$ because this is the only value of the adiabatic index that allows the bracket to satisfy the Jacobi identity, as shown by a direct proof of the Jacobi identity using the techinques of Ref. [82]. The Jacobi for this case will become evident in Sec. 3.6.

3.5 Casimir Invariants

As was mentioned in Section 2.2 one of the most important properties of noncanonical Hamiltonian systems is the existence of Casimir invariants, that is, constants of motion for any choice of Hamiltonian. A Casimir invariant

C needs to satisfy the relation $\{F, C\} = 0$ for any field F. Here, we will set $\partial_z = 0$. The generalization is straightforward.

Assuming a Casimir functional $C(n_i, M_i, T, n_e, M_e)$ and applying the condition $\{\xi_j, C\} = 0$ with $\xi_1 = n_i$, $\xi_2 = M_i$, $\xi_3 = T$, $\xi_4 = n_e$, $\xi_5 = M_e$ gives the following:

$$[n_i - 2u_d x, C_{n_i}] + [M_i, C_{M_i}] + [T - 2u_d x, C_T] = 0 \tag{3.21}$$

$$[n_i - 2u_d x, C_{M_i}] + [M_i, C_{n_i}] + [T - 2u_d x, C_{M_i}] + [M_i, C_T] = 0 \tag{3.22}$$

$$[n_i - 2u_d x, C_T] + [T - 2u_d x, C_{n_i}] + [M_i, C_{M_i}] = 0 \tag{3.23}$$

$$[n_e - 2u_d x, C_{n_e}] + [M_e, C_{M_e}] = 0 \tag{3.24}$$

$$\mu[n_e - 2u_d x, C_{M_e}] + [M_e, C_{n_e}] = 0\,. \tag{3.25}$$

For the rest of this section, we employ the previously defined variables $\tilde{n}_i$, $\tilde{n}_e$, $\tilde{T}$. In addition, we observe that $F_{\tilde{\xi}} = F_\xi$. From (3.21) and (3.25) we retrieve no information since they are automatically satisfied for any choice of C. However, from (3.22) we get

$$\begin{aligned}&[\tilde{n}_i, M_i](C_{M_i M_i} - C_{n_i n_i} - C_{T n_i})\\ +&[M_i, \tilde{T}](C_{n_i T} - C_{M_i M_i} + C_{TT})\\ +&[\tilde{n}_i, \tilde{T}](C_{M_i T} - C_{M_i, n_i}) = 0\,,\end{aligned} \tag{3.26}$$

from (3.23) we get

$$\begin{aligned}&[\tilde{n}_i, \tilde{T}](C_{TT} - C_{n_i n_i})\\ +&[\tilde{T}, M_i](C_{n_i,M_i} - C_{M_i T})\\ +&[\tilde{n}_i, M_i](C_{TM_i} - C_{M_i n_i}) = 0\,,\end{aligned} \tag{3.27}$$

and from (3.25) we get

$$[\tilde{n}_e, M_e](\mu C_{M_e M_e} - C_{n_e n_e}) = 0\,. \tag{3.28}$$

Accordingly, we have the following set of equations:

$$C_{M_i M_i} - C_{n_i n_i} - C_{T n_i} = 0 \tag{3.29}$$

$$C_{M_i M_i} - C_{n_i T} - C_{TT} = 0 \tag{3.30}$$

$$C_{TM_i} - C_{M_i n_i} = 0 \tag{3.31}$$

$$C_{TT} - C_{n_i n_i} = 0 \tag{3.32}$$

$$\mu C_{M_e M_e} - C_{n_e n_e} = 0\,, \tag{3.33}$$

which must be satisfied by any Casimir invariant.

We start from Eq.(3.31) and integrate it w.r.t M_i to find $C_{n_i} = C_T + f(\tilde{n}_i, \tilde{T})$. By using the method of characteristics on this result, we infer that the solution has the form $C = \Big\langle g(\tilde{T} + \tilde{n}_i, M_i) + f(\tilde{n}_i, \tilde{T}) \Big\rangle$, where the $\langle\rangle$ symbol implies an integral over the volume of interest. Subsequently, we substitute this form of the Casimir into (3.32) to obtain the wave equation $\partial^2_{n_i}(f+g) - \partial^2_T(f+g) = 0$ and by application of the method of characteristics, we recover the other characteristic direction, $C = \Big\langle g(\tilde{T} + \tilde{n}_i, M_i) + f(\tilde{n}_i - \tilde{T}) \Big\rangle$. Finally,

employing (3.29) we arrive at the wave equation $\partial^2_{M_i} g - 2\partial^2_{n_i+T} g = 0$. Invoking the method of characteristics once more, we derive the following general form for the Casimir invariants corresponding to the ion piece of the bracket:

$$C_i = \int d^2\mathrm{x} \quad g_\pm(\tilde{T} + \tilde{n}_i \pm \sqrt{2}M_i) + f(\tilde{n}_i - \tilde{T})\,. \tag{3.34}$$

For the Casimir invariants that correspond to the electron part of the bracket, we need only solve (3.33) to obtain

$$C_e = \int d^2\mathrm{x} \quad h_\pm(M_e \pm \sqrt{\mu}\tilde{n}_e)\,. \tag{3.35}$$

Thus, a general family of Casimir invariants is given by

$$\begin{aligned} C(n_i, M_i, T, n_e, M_e) = \int d^2\mathrm{x} \quad & g_\pm(\tilde{T} + \tilde{n}_i \pm \sqrt{2}M_i) \\ & + f(\tilde{n}_i - \tilde{T}) + h_\pm(M_e \pm \sqrt{\mu}\tilde{n}_e)\,, \end{aligned} \tag{3.36}$$

where $g_\pm$, f and $h_\pm$ are arbitrary functions.

3.6 Normal Fields

The general form of the Casimir (3.36) suggests the introduction of a new set of variables which are called "normal fields" (see e.g. Refs. [126, 123, 122]):

$$\mathcal{V}_{i,\pm} = \tilde{T} + \tilde{n}_i \pm \sqrt{2}M_i \tag{3.37}$$

$$\mathcal{V}_{i,f} = \tilde{n}_i - \tilde{T} \tag{3.38}$$

$$\mathcal{V}_{e,\pm} = M_e \pm \sqrt{\mu}\tilde{n}_e\,. \tag{3.39}$$

We claim that if we express the equations of motion (3.10) – (3.14) and the bracket of (3.20) in terms of these fields, they will take a simple form. To do so, the following chain rule expressions for functional derivatives in terms of these new fields are required:

$$F_{n_i} = F_{\mathcal{V}_{i,+}} + F_{\mathcal{V}_{i,f}} + F_{\mathcal{V}_{i,-}} \tag{3.40}$$

$$F_{T_\parallel} = F_{\mathcal{V}_{i,+}} + F_{\mathcal{V}_{i,-}} - F_{\mathcal{V}_{i,f}} \tag{3.41}$$

$$F_{M_i} = \sqrt{2}\left(F_{\mathcal{V}_{i,+}} - F_{\mathcal{V}_{i,-}}\right) \tag{3.42}$$

$$F_{M_e} = F_{\mathcal{V}_{e,+}} + F_{\mathcal{V}_{e,-}} \tag{3.43}$$

$$F_{n_e} = \sqrt{\mu}\left(F_{\mathcal{V}_{e,+}} - F_{\mathcal{V}_{e,-}}\right) . \tag{3.44}$$

Using (3.40)-(3.44) the Poisson bracket of (3.20) becomes

$$\begin{aligned}\{F,G\} = &-2\big\langle \mathcal{V}_{i,f}[F_{\mathcal{V}_{i,f}}, G_{\mathcal{V}_{i,f}}] \\ &+ 2\left(\mathcal{V}_{i,+}[F_{\mathcal{V}_{i,+}}, G_{\mathcal{V}_{i,+}}] + \mathcal{V}_{i,-}[F_{\mathcal{V}_{i,-}}, G_{\mathcal{V}_{i,-}}]\right) \\ &- \sqrt{\mu}\left(\mathcal{V}_{e,+}[F_{\mathcal{V}_{e,+}}, G_{\mathcal{V}_{e,+}}] - \mathcal{V}_{e,-}[F_{\mathcal{V}_{e,-}}, G_{\mathcal{V}_{e,-}}]\right) \\ &+ 2\sqrt{2}\left(F_{\mathcal{V}_{i,+}}\partial_z G_{\mathcal{V}_{i,+}} - F_{\mathcal{V}_{i,-}}\partial_z G_{\mathcal{V}_{i,-}}\right) \\ &- \sqrt{\mu}\left(F_{\mathcal{V}_{e,+}}\partial_z G_{\mathcal{V}_{e,+}} - F_{\mathcal{V}_{e,-}}\partial_z G_{\mathcal{V}_{e,-}}\right)\big\rangle .\end{aligned} \tag{3.45}$$

This simple form of the bracket is called a direct product [126], and its form immediately ensures the Jacobi identity. Since the inner brackets satisfy the Jacobi identity, so do their sums which constitute the larger bracket of Eq.(3.20).

Having expressed the bracket in terms of the normal fields, we can now

write down the equations of motion that these fields satisfy, viz.

$$\frac{\partial \mathcal{V}_{i,\pm}}{\partial t} + [\mathcal{A}_{i,\pm}, \mathcal{V}_{i,\pm}] \pm \sqrt{2}\partial_z \mathcal{A}_{i,\pm} = 0 \tag{3.46}$$

$$\frac{\partial \mathcal{V}_{e,\pm}}{\partial t} + [\mathcal{A}_{e,\pm}, \mathcal{V}_{e,\pm}] \mp \sqrt{\mu}\partial_z \mathcal{A}_{e,\pm} = 0 \tag{3.47}$$

$$\frac{\partial \mathcal{V}_{i,f}}{\partial t} + [\mathcal{A}_{i,f}, \mathcal{V}_{i,f}] = 0\,, \tag{3.48}$$

where

$$\mathcal{A}_{i,\pm} = \Phi + n_i + T \pm \sqrt{2}u_i \tag{3.49}$$

$$\mathcal{A}_{i,f} = \Phi + n_i - T \tag{3.50}$$

$$\mathcal{A}_{e,\pm} = \pm\left(\frac{n_e}{\tau} - \phi\right) + \mu^{\frac{3}{2}}u_e \tag{3.51}$$

are stream functions that simply convect the fields $\mathcal{V}_{s,\pm/f}$. The latter are therefore Lagrangian conserved quantities. Note that in a turbulent system, equipartition results in the flattening of the profiles of Lagrangian invariants.[96]

3.7 Linear Study

In this section we linearize (3.10)–(3.14) and the two closure relations (3.16)–(3.17) about an inhomogeneous equilibrium configuration. Then, after deriving the dispersion relation, we study the linear stability of the ITG mode. We assume that the densities and temperature vary linearly in the x direction, i.e., that these quantities have the form $f = x/L_f + \delta f$ with $\delta f = \hat{f}\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$. This may be interpreted as a local study, in the WKB sense, for modes satisfying $k_\perp L_\perp \gg 1$ and $k_\parallel L_\parallel \gg 1$, where $L_\perp$ and $L_\parallel$ represent equilibrium scale-lengths. Our purpose is to obtain some physical understanding of our

model and see how accurately it can describe the various modes of interest. Next, we assume $\Phi_{eq} = 0$ and $\nabla\psi_{eq} \times \hat{z} = B_{oy}\hat{y}$ with $B_{oy} = -\frac{\partial\psi}{\partial x}$ a constant and $u_{i,eq} = 0$. We note that in Fourier space, the operator Γ_o is $\Gamma_o(b) = e^{-b}I_o(b)$, where $b = k_\perp^2\rho_i^2$ (or $b \equiv k_\perp^2$ in our normalized units). Even though we mentioned that the model is Hamiltonian only for the choice $\gamma = 2$, in the following we keep γ general to investigate its effect on the behavior of the modes and we subsequently set $\gamma = 2$, to recover the results for our model.

Moreover, we add two dissipative terms to Eq. (3.12) that are related to the parallel and toroidal resonances. Therefore, from now on, we make the distinction between the non-dissipative, i.e. Hamiltonian, gyrofluid model and the one where dissipation terms are included.

Parameters χ and ν of the added dissipative terms are tuned so that the response function of a gyrofluid model matches the kinetic one in the slab and the toroidal limits, respectively. Their values have been computed in Refs. [34, 136] and found to be $\chi = \frac{2}{\sqrt{\pi}}$ and $\nu = 2.019$. Although the χ value is exact, the numerical value of ν has not been calculated for the particular model we are presenting but for a similar gyrofluid model. Nevertheless, we will adopt it. The reason is that here, we are mainly concerned with the non-dissipative, Hamiltonian part of the model and the addition of the dissipative terms is not intended to enhance the accuracy of the results, but merely to show the reader that such a modification is indeed possible. Correct treatment of dissipation would require the proper study of the response function of a kinetic model containing the same physics and the numerical minimization of

the error in matching it with the response function obtained by (3.10)–(3.14). Such a study is beyond the goals of this dissertation.

The linearization of the equations of motion and the closure relations in Fourier space result in the following system of equations:

$$-\omega\hat{n}_i = \omega_*\Gamma_0^{1/2}(b)\hat{\phi} - k_y B_{oy}\hat{u}_i - 2\omega_*\epsilon(\hat{n}_i + \Gamma_0^{1/2}(b)\hat{\phi} + \hat{T}) - k_z\hat{u}_i\,, \tag{3.52}$$

$$\begin{aligned}-\omega(\Gamma_0^{1/2}(b)\hat{\psi} + \hat{u}_i) =& - k_y B_{oy}\hat{T} - \omega_*\eta_i\Gamma_0^{1/2}(b)\hat{\psi} - k_y B_{oy}\Gamma_0^{1/2}(b)\hat{\phi} - k_y B_{oy}\hat{n}_i \\ & - \omega_*\Gamma_0^{1/2}(b)\hat{\psi} - 4\omega_*\epsilon\hat{u}_i - k_z\hat{n}_i - k_z\hat{T} - k_z\Gamma_0^{1/2}(b)\hat{\phi}\,,\end{aligned} \tag{3.53}$$

$$\begin{aligned}-\omega\hat{T} =& \omega_*\eta_i\Gamma_0^{1/2}(b)\hat{\phi} - (\gamma-1)k_y B_{oy}\hat{u}_i - 2\omega_*\epsilon(\hat{n}_i + \Gamma_0^{1/2}(b)\hat{\phi} + \hat{T}) \\ & - (\gamma-1)k_z\hat{u}_i + 2i\nu|\omega_*|\epsilon\hat{T} + i\chi|k_\parallel|\hat{T}\,,\end{aligned} \tag{3.54}$$

$$-\omega\hat{n}_e = \omega_* r_n\hat{\phi} - k_y B_{oy}\hat{u}_e + 2\omega_*\epsilon\left(\frac{\hat{n}_e}{\tau} - \hat{\phi}\right) - k_z\hat{u}_e\,, \tag{3.55}$$

$$\begin{aligned}-\omega(\hat{\psi} - \mu\hat{u}_e) =& - k_y B_{oy}\hat{\phi} + \frac{\omega_* r_n}{\tau}\hat{\psi} + \frac{k_y}{\tau}B_{oy}\hat{n}_e \\ & + 2\omega_*\epsilon\mu\hat{u}_e - k_z\hat{\phi} + \frac{k_z}{\tau}\hat{n}_e\,,\end{aligned} \tag{3.56}$$

$$\hat{n}_e = \Gamma_0^{1/2}(b)\hat{n}_i + (\Gamma_0(b) - 1)\hat{\phi}\,, \tag{3.57}$$

$$\frac{2}{\tau\beta_e}k_\perp^2\hat{\psi} = - \hat{u}_e + \Gamma_0^{1/2}(b)\hat{u}_i\,. \tag{3.58}$$

Note that $\Gamma_0^{1/2}(b)B_{oy} = B_{oy}$ and, to be clear, recall the ion and electron density and parallel temperature gradients vary linearly, i.e., $n_i = x/L_{n_i}, n_e = x/L_{n_e}$, and $T = x/L_T$. We simplify the result by setting $k_\parallel = k_z + B_{oy}k_y$ and by defining the parameters $\eta_i = L_{n_i}/L_T$, $\epsilon = u_d L_{n_i}$, and $r_n = L_{n_i}/L_{n_e}$. Also, $\hat{\omega}_* = (cT_e/eB_o)(\hat{k}_y/L_n)$ is the usual diamagnetic frequency. In dimensionless variables it is expressed as $\omega_* = \tau u_{ti}k_y/L_n$.

3.7.1 Electrostatic Dispersion Relation

The electrostatic limit, which is applicable for low-β conditions, [53] leads to a cubic dispersion relation that offers the opportunity of comparing analytic solutions of the gyrofluid model to kinetic results. To make contact with well-known analytic results for the slab branch of the ITG mode, we also neglect toroidal effects. That is, we drop all toroidal terms of Eqs. (3.52)–(3.56), set $\hat{\psi} = 0$, and study the slab, electrostatic ITG modes, where the drive is due to the coupling of the parallel transit of particles with the temperature gradient. We notice that in this case, the electron and ion fields are decoupled so we only use the ion fields of Eqs. (3.52)–(3.54), along with the quasineutrality condition of (3.58) and the electron adiabatic response $n_e \approx \phi/\tau$[1]. After straightforward manipulations, we obtain a dispersion relation with real part given by

$$\left(\frac{1}{\tau} + 1 - \Gamma_o(b)\right)\omega^3 + \gamma k_\parallel^2 \left(\Gamma_o(b)\frac{\gamma - 1}{\gamma} - \frac{1}{\tau} - 1\right)\omega \\ - \Gamma_o(b)\omega_*\omega^2 + \Gamma_o(b)k_\parallel^2\omega_*\left((\gamma - 1) - \eta_i\right) = 0 \tag{3.59}$$

and imaginary part by

$$k_\parallel \left(\frac{1 + (1 - \Gamma_o(b))\tau}{\tau}\omega^2 + \Gamma_o(b)\omega_*\omega - \frac{k_\parallel^2(1+\tau)}{\tau}\right) = 0\,. \tag{3.60}$$

Returning to Eq. (3.59) we can infer two stability criteria. The first one comes from neglecting the dissipative terms, hence having just the real

[1]Indeed, this is a very strong assumption as can be seen in Ref. [52] where, retaining $[n, \phi]$ terms results in a much better estimate of the drift wave fluctuation level compared to the one of Hasegawa-Mima equation.

part of the dispersion relation and by demanding the third-order polynomial to have only real roots. This is done by setting the cubic discriminant equal to zero and by that deriving a quadratic equation in η_i. To investigate the case of finite $k_\perp$, we obtain the stability criterion by making no approximation on $\Gamma_o(b)$. The result is shown in Fig. 3.4 where η_{crit} (the root of the quadratic equation mentioned above) has been plotted as a function of b for various values of $k_\parallel$. The curves depicted in Fig. 3.4 are qualitatively similar to those

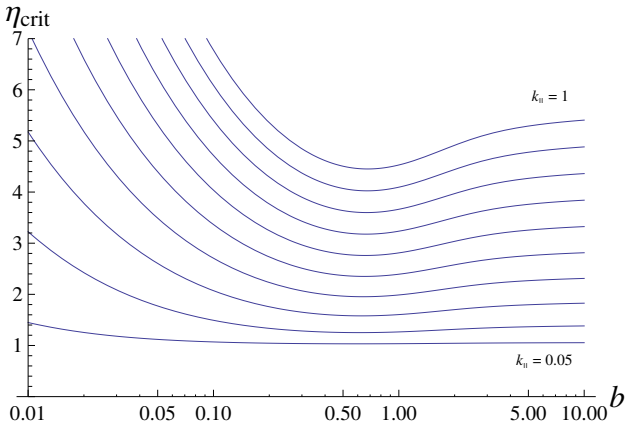


Figure 3.4: Stability criterion with finite $k_\perp$ as given by b. Here $k_\parallel$ ranges from 0.05 to 1.0

reported in Ref. [5] where a kinetic model was used.

The second stability criterion we deduce, concerns the case of perturbations with very long parallel wavelengths and comes from setting the imaginary part of the dispersion relation equal to zero, solving for ω under the condition $k_\parallel = 0$, and eliminating it from Eq. (3.59). With this procedure we find

$$\eta_{crit}^{GF} = \gamma - 1\,. \tag{3.61}$$

Observe, the critical value depends on the adiabatic index. The kinetic result for this limiting case is provided in Ref. [58] and is given by

$$\eta_{crit}^{KIN} = \frac{2}{1 + 2b\left(1 - \frac{I_1(b)}{I_o(b)}\right)}, \tag{3.62}$$

with $I_1(b)$ and $I_0(b)$ being modified Bessel functions of the first kind. Note that the adiabatic index in the exact moment equation for the evolution of the parallel temperature is 3. In Fig. 3.5 we plot this relation and the corresponding fluid approximation of it and we notice that our gyrofluid model has the correct asymptotic behavior for perturbations with very small perpendicular wavelengths provided $\gamma = 2$. However, had we chosen $\gamma = 3$, we would have gotten the correct asymptotic behavior for very large perpendicular wavelengths, at the cost of a non-Hamiltonian model. Moreover, the choice $\gamma = 5/3$ gives $\eta_{crit} = 2/3$, the result for the fluid model of Ref. [5].

The reason behind this discrepancy stems from the fact that our model lacks an equation for the evolution of the perpendicular temperature. Therefore, all assumptions about the correlation of $T_\perp$ and $T_\parallel$ are made by the choice of γ (with $\gamma = 3$ meaning $T_\perp$ and $T_\parallel$ are uncorrelated and $\gamma = 5/3$ meaning $T_\perp = T_\parallel$) and remain fixed throughout the dynamics. Despite this obvious inflexibility of the gyrofluid model, it is evident from Fig. 3.5 that it still remains far superior compared to its FLR counterpart.

It is helpful to study the 'fluid' limit of Eq. (3.59), which is obtained by setting $\Gamma_o(b) = 1$ corresponding to $b = 0$. This is the limit of very long perpendicular wavelengths compared to the gyroradius. Figure 3.6 shows the

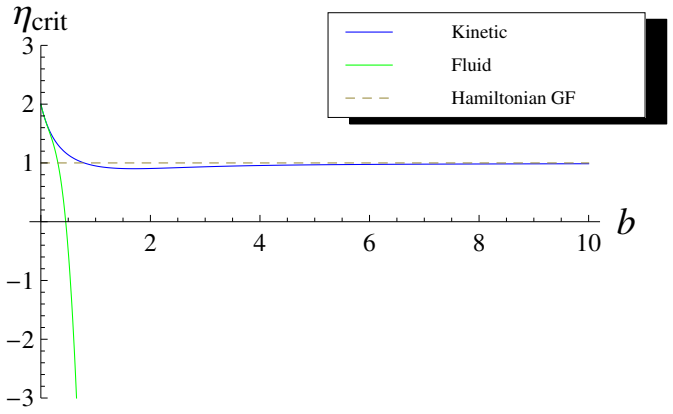


Figure 3.5: Comparison of critical η for kinetic, fluid and gyrofluid results for the case $k_{\|} = 0$

stability criterion in this fluid limit for three different values of γ, results that were previously obtained in Ref. [5] for $\gamma = 5/3$, where a heuristic explanation was given for the η_{crit} limiting value for very long $k_{\|}$.

To conclude with the electrostatic slab case, we investigated the growth rate as a function of τ. The condition $\tau < 1$ or, in other words, $T_i > T_e$ is a well-known stabilizing factor for ITG, which is of particular importance for the hot-ion cores of tokamaks.[33, 107, 24] Indeed, the behavior we found was the expected one as we can see in Fig. 3.7. It is believed that the reason for the stabilizing effect of high T_i is that it weakens the density perturbation and because of that, the potential response, weaking in this fashion the feedback mechanism of ITG.

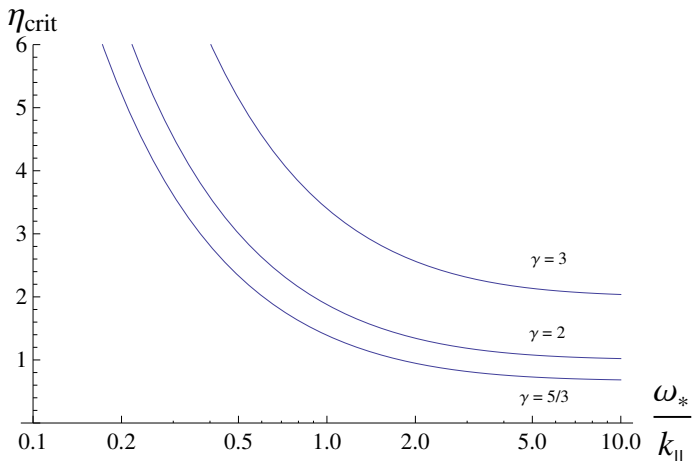


Figure 3.6: Stability criterion at the 'fluid' limit with $\tau = 1$ for different values of the adiabatic index

3.7.2 Electromagnetic Dispersion Relation

To be applicable to the higher plasma pressure achieved by auxiliary or alpha-particle heating, the theory must include the electromagnetic effect. In fact, this effect becomes important at surprisingly low-β because of other small parameters in the problem. It is well known that increasing β stabilizes ITG modes [62], but leads to the onset of kinetic ballooning modes, also known as the Alfvénic ITG modes (AITG).[144] For toroidal ITG modes, the drive comes from the coupling of curvature and ∇B-drift terms with the temperature gradient, so that we must now keep the toroidal curvature terms. It can be easily seen that, to lowest order, the electromagnetic effect is stabilizing. The electromagnetic perturbation creates a small component of $\mathbf{B}$ that is perpendicular to both the background magnetic field and the pressure gradient.

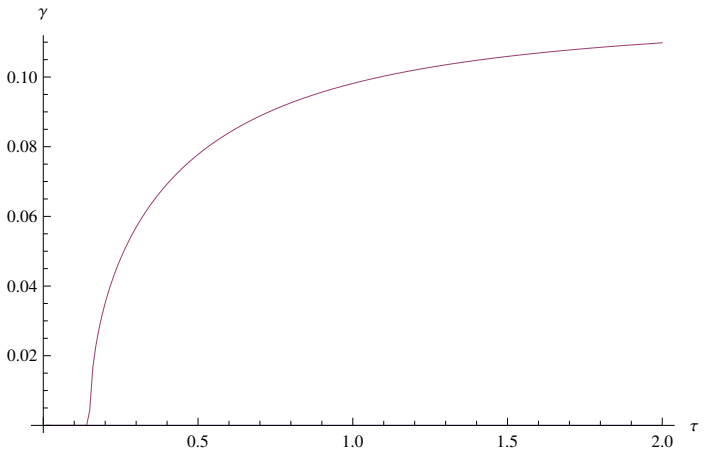


Figure 3.7: Normalized growth rate as a function of τ for $\eta_i = 2.5, b = 0.5, k_{\|} = 0.1$

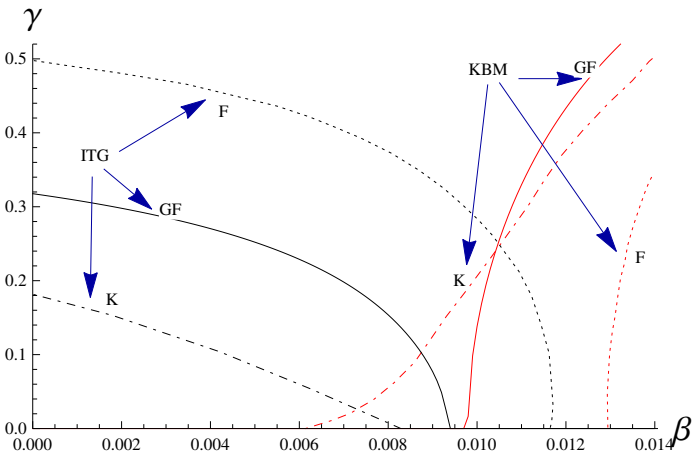
This component then leads to the development of a force on the ions, parallel to the field lines that opposes the attraction from the pressure lowering of ITG.

In the remainder of this section, we follow the analysis of Kim, Horton and Dong[62] and compare our gyrofluid results with their local kinetic ones. We note, however, that complete agreement cannot be expected since Kim *et al.* has one extra parameter, namely η_e. We also note that the eigenfrequencies for the model in Ref. [113] lie within a few percent of the kinetic results, so that comparing our model to the kinetic results is effectively equivalent to comparing it to the Snyder and Hammett model.

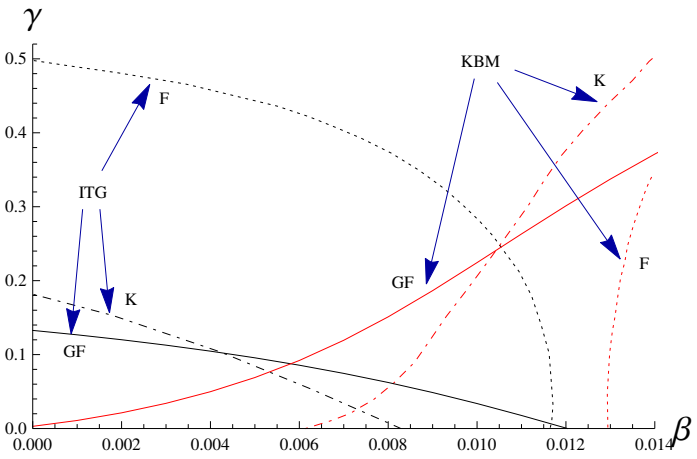
Because the dispersion relation becomes unwieldy and doesn't provide much physical insight, we refrain from displaying it here. Instead, we solve it numerically and present the results. Figure 3.8 shows the normalized growth

rates for (a) the ideal and (b) the "Landau" versions of our model as a function of β when $\eta_i = 2.5, b = 0.5, k_\parallel = 0.1, \epsilon = 0.2, r_n = 1$ and $\tau = 1$. We also provide the kinetic and fluid model results from Ref. [62] for comparison. By the "Landau" version we mean of course the Hamiltonian model augmented by dissipative terms modeling the damping caused by the wave-particle interactions. From Fig. 3.8a, it becomes immediately clear that the nonlocal treatment of the ion response in the gyrofluid model reproduces the main qualitative features of the kinetic result much better than the fluid model. Compared to the fluid result, the gyrofluid one gives stronger stabilization of the ITG modes and lower thresholds for the excitation of KBMs. This is related to the toroidal resonance. In both Fig. 3.8a and Fig. 3.8b we observe the close connection between the stabilization of the ITG mode and the excitation of the Kinetic Ballooning mode in accordance with what kinetic theory predicts. The addition of dissipative terms makes the curves shift closer to the kinetic result, although we remark that at low growth rates the agreement is less satisfactory.

Here, we pause to explain an interesting effect, the destabilization due to the addition of dissipation of two previously marginally stable modes ($\gamma = 0$). For example, for the GF model it is seen in Fig. 3.8a that without dissipation the KBM becomes unstable at $\beta \approx 0.010$, while in Fig. 3.8b it is seen for the same case with dissipation that this mode is destabilized for all values of β. A similar shift from stability to instability can be observed upon comparing these figures for the ITG mode, which is seen in Fig. 3.8a to tran-



(a) Hamiltonian model without dissipative terms and comparison with kinetic and fluid results.



(b) Model with dissipative terms and comparison with kinetic and fluid results.

Figure 3.8: Normalized growth rate vs. β for $\eta_i = 2.5, b = 0.5, k_\parallel = 0.1, \epsilon = 0.2, r_n = 1$, and $\tau = 1$. Toroidal ITG is the black line while KBM is the red.

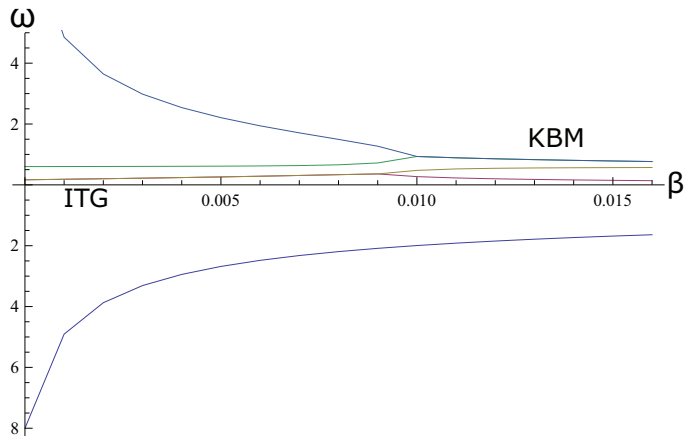


Figure 3.9: Real frequency vs. β. All parameters are the same as in Fig.4

sition to instability at a somewhat smaller value of β. To understand these transitions consider Fig. 3.9, where we plot the real parts of the frequencies versus β for four modes of the GF model without dissipation. In this figure the two upper most modes are marginally stable for small values of β, then as β approaches the transition value near 0.010 they collide and produce instability. This bifurcation, which is standard in Hamiltonian systems, is called the Hamiltonian Hopf (or Kreĭn) bifurcation [83, 88]. Observe, the same bifurcation occurs when two marginally stable modes collide as β is decreased to a value near to the KBM transition but closer to 0.009, producing the unstable ITG mode. (After the transitions there are also damped modes that are not shown in the figures.)

We can observe the Kreĭn bifurcation in Fig. 3.10 where we have plotted the complex frequency of the two modes of Fig. 3.9 that bifurcate to give the KBM mode. They correspond to the blue and green lines in the upper

half plane of Fig. 3.9. The two modes are shown in blue and red, the y-axis corresponds to the imaginary part (growth rate) and the x-axis to the real part (real frequency) of the modes complex frequency. In Fig. 3.10(a) we see initially both modes are stable. As we let the parameter β increase, Fig. 3.10(b), they remain stable however the "blue" mode changes real frequency and approaches the "red" one. As β increases further, the two modes reach a point where they are about to collide. This situation is depicted in Fig. 3.10(c). Any further increase in β from that point on will result in the bifurcation of the two modes. What happens is that the modes develop an imaginary component with the imaginary part of one being the complex conjugate of the imaginary part of the other. Indeed, this case is portrayed in Fig. 3.10(d) where the "red" mode has become a damped mode and the "blue" mode has been destabilized. Actually, the "blue" mode now, corresponds to the unstable KBM mode. In Fig. 3.10 we are plotting the value of the complex frequencies for all values of the changing β so one can see more clearly the paths that the modes trace before, during and after the bifurcation.

The dissipative destabilization observed in Fig. 3.8b is a generic feature of Hamiltonians systems with negative energy modes (NEMs). Indeed, in Hamiltonian systems, whenever a Hamiltonian Hopf bifurcation occurs, one of the modes *must* be a NEM, and such modes have the property of getting destabilized with the addition of dissipation (see e.g. Ref. [90] for a Hamiltonian version of the classical Kelvin-Tait theorem[127]). One could perform a calculation like those of Refs. [123, 121], where the modal eigenvector is in-

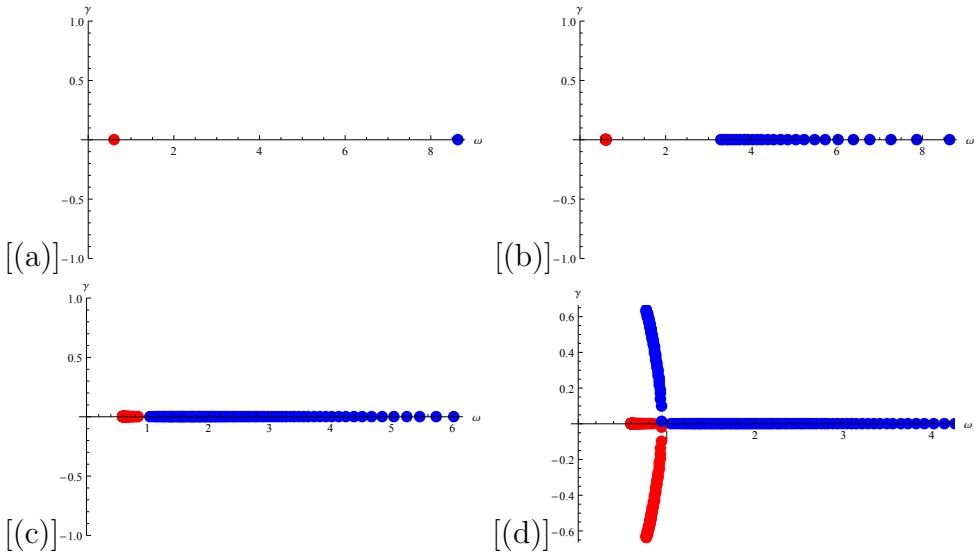


Figure 3.10: Kreĭn bifurcation

serted into the perturbation energy in order to show explicitly that it is an NEM and then show the dissipative terms remove energy from this mode, but such a calculation is outside the scope of the present chapter. (A similar situation occurs when energy is added to a positive energy mode.) Also note, the previously unstable ITG and KBM modes of Fig. 3.8a become less unstable at the onset of dissipation, as is shown in Fig. 3.8b due to the fact that it becomes harder for a mode to grow when there is less available energy in the system, which is consistent with this scenario.

We reiterate that the purpose of our model is to improve the *nonlinear* fidelity of fluid models. From that perspective, we view the quality of agreement in Fig. 3.8 as adequate.

In Fig. 3.11 we display the dependence of the growth rates of the ITG

and KBM modes on $k_\parallel$ for various values of β for the model augmented with the dissipative terms, with all other parameters remaining the same as in Fig. 3.8. Values of β are in the range $0.000 - 0.012$. We observe that the stabilization through the electromagnetic effect becomes more efficient with decreasing $k_\parallel$. Further, we see again the near simultaneous stabilization of ITG and destabilization of KBM as was noted above.

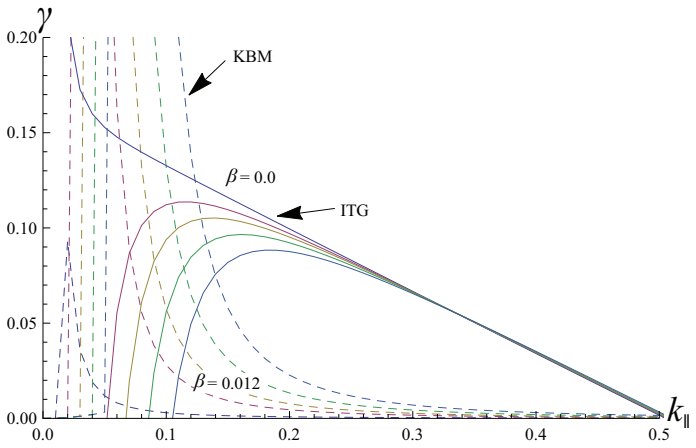


Figure 3.11: Growth rates of the ITG-KBM modes as a function of the parallel wavenumber for the gyrofluid model with dissipative terms.

For large values of $k_\parallel$ the mode is stabilized by the large parallel ion transit term.[24, 61, 23] Intuitively, we can understand that the mode is limited by the fact that an appreciable initial density perturbation cannot be created within an arbitrarily small length scale. Even before this limit is reached, though, the negative compressibility mentioned above is proportional to the ratio of the ion diamagnetic to the sound frequency ($\omega_{pi}/k_\parallel c_s$), so that coupling to the sound wave acts as a source of stabilization. On the other hand,

the initial density and potential perturbations, as well as the resulting pressure lowering, are all proportional to $k_\parallel$. Therefore, a finite $k_\parallel$ is needed to overcome the stabilizing effect of curvature and β. Thus, the mode becomes most unstable at some intermediate value.

To explain the above in physical terms, we observe that the perturbations' $k_\parallel$ is set by the competition of Landau Damping and phase mixing with "good"/"bad" curvature effects. The perturbations tend to be aligned with the field lines. This is because Landau Damping and phase mixing along the field line quickly kills any high $k_\parallel$ component. On the other hand, as the field line rotates it samples both "good" and "bad" curvature regions. This results in increased perturbation amplitudes in the bad curvature regions and reduced amplitudes in good curvature regions. This phenomenon is called localization and the resulting mode structure is known as ballooning mode. The interplay between these two competing effects creates modes with $k_\parallel = \frac{1}{qR}$ where q is the safety factor and R the major radius of the tokamak [8].

We remark that in practice a complete treatment of the effect of $k_\parallel$ requires a nonlocal approach since $k_\parallel L_\parallel \sim 1$ normally applies, so that the WKB approach in insufficient.

In Fig. 3.12 we illustrate the behavior of the growth rate versus $k_\perp$ for various values of β, with the same parameters as those of the previous figures. We notice that the peak growth rate occurs around $k_\perp \approx 0.65$ and does not change much with β. Furthermore, the stabilizing effect of β is almost uniform for wavenumber values higher than this. This could be attributed to a

very high phase velocity of the wave, which leaves few particles with the right thermal speed to resonate with it. For smaller wavenumbers, however, the stabilization due to β becomes ineffective. This is because in this region the parallel ion transit term becomes significantly larger than the curvature term and becomes the dominant stabilizing effect. Another important stabilizing effect at high $k_\perp$ comes from FLR physics. Namely, the ions respond to the gyroaveraged electrostatic field, thereby reducing the effective $E \times B$ velocity.

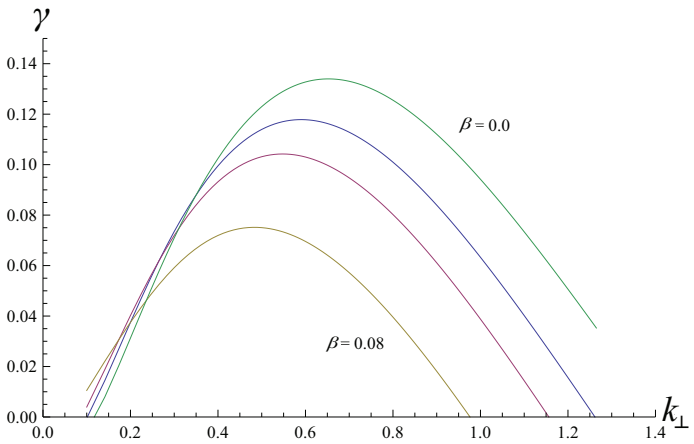


Figure 3.12: Growth rates of the ITG-KBM modes as a function of the perpendicular wavenumber for the gyrofluid model with dissipative terms.

In Figs. 3.13 and 3.14 we compare the Hamiltonian model augmented by dissipative terms and the kinetic result from Ref. [62]. Both figures suggest some common features: again, the qualitative similarity between the Hamiltonian and the kinetic curves is evident. However, there is a quantitative disparity since the Hamiltonian result is roughly three times higher than the kinetic one at the peak value of γ. This deviation seems to be corrected by

taking into account the dissipative terms which lowers the results to at most 30% off from the kinetic ones at peak growth rate. This amendment, though, doesn't come without its own problems, namely the erratic behavior of the dissipative model at low values of γ.

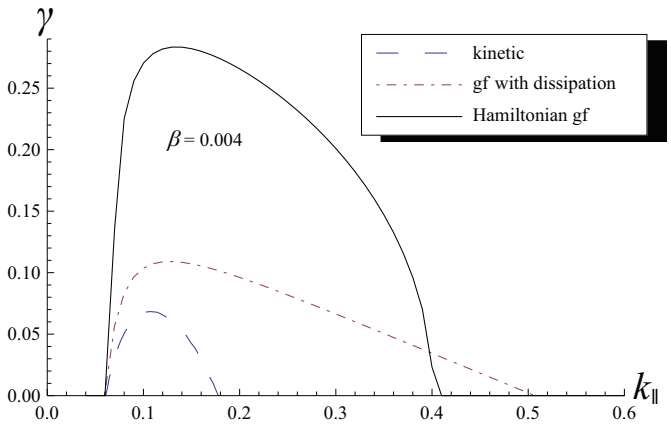


Figure 3.13: Comparison between growth rates of the ITG mode as a function of the parallel wavenumber for the ideal Hamiltonian model, the model with linear wave-particle (Landau) damping, and the kinetic results.

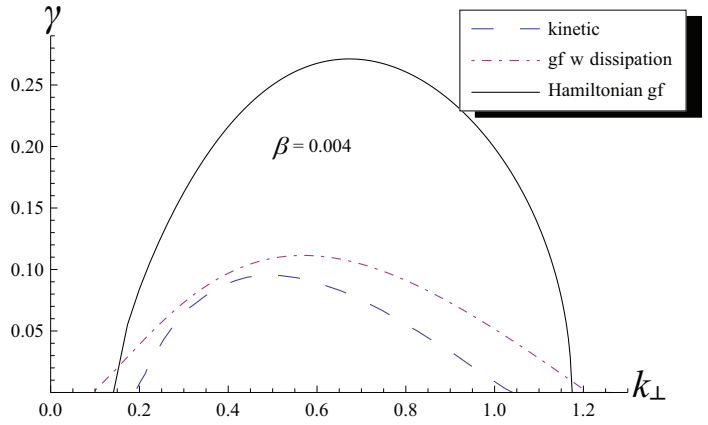


Figure 3.14: Comparison between growth rates of the ITG mode as a function of the perpendicular wavenumber for the ideal Hamiltonian model, the model with linear wave-particle (Landau) damping, and the kinetic results.

Chapter 4

Action Principles in Fluids and Plasmas

The second part of this dissertation presents a general method of arriving at Hamiltonian reduced fluid models via the use of an action principle. Before we start this undertaking in the next chapter, we believe it is appropriate to remind the reader about Hamilton's Principle of least action, variational derivatives and the two equivalent descriptions of fluid dynamics. All these are tools that we will use.

4.1 Hamilton's Principle of least action

In Chapter 2 we showed how one could arrive at Hamilton's equations using the Legendre transformation if one knows the Lagrangian of the system. In fact, the Hamiltonian description of dynamical systems is self-contained and Hamilton's equations, which are the equations of motion, follow naturally from the Hamiltonian function and the form of the Lie-Poisson bracket. Now, we are going to take the Lagrangian route where the equations of motion of the system are not given in terms of the phase space variables q and p but in terms of q and $\dot{q}$. What is needed to arrive at the dynamical equations, which in this context are called the Euler-Lagrange (E-L) equations, is an action

functional, $\mathcal{S}$. The definition of the action is the following:

$$\mathcal{S}[q^i] = \int_{t_1}^{t_2} dt\, \mathcal{L}(q^i, \dot{q^i}, t)\,, \tag{4.1}$$

where $\mathcal{L}$ is the Lagrangian of the system. The procedure one has to follow in order to obtain the E-L equations is to find the path $q(t)$ that extremizes $\mathcal{S}$ when the end points $q(t_1)$, $q(t_2)$ are fixed. This procedure amounts to finding the variational derivatives of the action with respect to q^i's and setting them equal to zero.

4.1.1 Variational Derivatives

Because the variational principle is the main tool of this and the next chapter, we will pause here to remind the reader about variational derivatives. Lets start with a functional $F[q]$ defined as follows:

$$F[q] = \int_{t_1}^{t_2} dt\, \mathcal{F}(q, \dot{q}, t)\,. \tag{4.2}$$

The first variation of F is the first-order change in F as $q(x)$ changes to $q(x) + \epsilon\delta q(x)$. Formally it can be written as

$$\begin{aligned} \delta F[q;\delta q] &= \frac{d}{d\epsilon} F\left[q + \epsilon\delta q\right]\Big|_{\epsilon=0} \\ &:= \int \delta q \frac{\delta F}{\delta q}\, dx := \left\langle \frac{\delta F}{\delta q}, \delta q \right\rangle. \end{aligned} \tag{4.3}$$

In the above definition, the isolated term $\dfrac{\delta F}{\delta q}$ is known as the variational derivative of F with respect to q.

The functional F depends on q and it's derivatives. This means that changes on F are not only induced by changes of q alone but also by changes in $\dot{q}$ and higher derivatives. Taylor expanding F around q and it's higher derivatives, keeping only first order terms and then taking the derivative with respect to ϵ, we are left with:

$$\delta F[q;\delta q] = \int \left(\frac{\partial \mathcal{F}}{\partial q}\delta q + \frac{\partial \mathcal{F}}{\partial q_t}\delta q_t + \frac{\partial \mathcal{F}}{\partial q_{tt}}\delta q_{tt} + \dots \right) dt \tag{4.4}$$

$$= \int \delta q \left(\frac{\partial \mathcal{F}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{F}}{\partial q_t} + \frac{d^2}{dx^2}\frac{\partial \mathcal{F}}{\partial q_{tt}} - \dots \right) dt\,, \tag{4.5}$$

where in the second line we have used integration by parts and the fact that the operations of derivation, $\frac{d}{dx}$, and variation, δ, commute.

Now, we are able to extremize the action of Eq. (4.1):

$$\delta \mathcal{S} = \int_{t_1}^{t_2} dt \, \left(\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) \delta q^i = 0\,, \tag{4.6}$$

where the surface terms arising from the integration by parts vanish owing to the fact that the end points are fixed, therefore any variation of the path, when evaluated on the end points must be zero. Because Eq. (4.6) needs to hold true for any variation of the path δq^i, the fundamental lemma of calculus of variations requires that:

$$\frac{\delta \mathcal{S}}{\delta q^i} = \frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0\,. \tag{4.7}$$

which are the celebrated Euler-Lagrange equations.

4.2 Lagrangian vs Eulerian Description

In this section we will introduce and juxtapose the two equivalent ways one can describe the motion of fluids. We will start with the lesser known of the two, the Lagrangian. In the Lagrangian description of fluids, we first introduce the concept of a fluid element or fluid particle. We assume that the fluid is comprised of infinitesimal elements, each of which retains all the properties of the original macroscopic fluid. Therefore, a complete (kinematic at least) description of the system can be accomplished if we know the positions of these fluid elements at all times. We define the position of a fluid element at time t to be given by the function $q(a, t)$. The domain of this function is the space of all fluid elements, therefore, the variable a must somehow specify which element's position we want to know. The variable a serves as the fluid element label or fluid label for the sake of brevity. In general, there are various labeling schemes we can follow in order to uniquely specify all the fluid elements. A straightforward one is to label each element by the three coordinates of its initial position. Formally, $a = (a^1, a^2, a^3) = q(a, 0)$. In this way, the domain of the function q becomes D, the domain of the fluid and since, naturally, the range of q is the same, we can say that q is a $D \to D$ map. Furthermore, unless we don't have to deal with pathological situations where fluid elements overlap, this map is also invertible. Because a fluid is a continuous medium and because we can (ideally) choose our fluid elements to be infinitesimally small, given the labeling scheme we just described, the label a becomes a continuous label. Thus, it makes sense to define a deformation

matrix $q^i_{,j} := \dfrac{\partial q^i}{\partial a^j}$ which is of course the Jacobian matrix of the map that sends fluid elements to their positions in space. We reserve the term "Jacobian" to refer only to the determinant of $q^i_{,j}$. Now, the statement of invertibility of q becomes a statement on the Jacobian, namely, $\mathcal{J} = \det(q^i_{,j}) \neq 0$. In case this is true, and for the rest of this and the next chapter we will suppose that there are no problems with the invertibility of q, the deformation matrix has an inverse $a^i_{,j} = \frac{\partial a^i}{\partial q^j}$. It follows then that,

$$q^i_{,k} a^k_{,j} = a^i_{,k} q^k_{,j} = \delta^i_{,j} \,. \tag{4.8}$$

It will be useful and instructive to write the above equation in terms of the adjoint matrix. Here, we need to remind the reader that the definition of the adjoint of a matrix is that it is the transpose of its cofactor matrix. In that case, we define the adjoint of $q^i_{,j}$ to be $A^j_{,i}$. According to Crammers rule, the inverse of a matrix is the matrix adjoint divided by the determinant. From this Eq. (4.8) can be written as:

$$q^i_{,k} \frac{A^k_{,j}}{\mathcal{J}} = \delta^i_{,j} \,. \tag{4.9}$$

In three dimensions, $\mathcal{J}$ can be calculated as:

$$\mathcal{J} = \frac{1}{6} \epsilon_{kjl} \epsilon_{imn} q^k_{,i} q^j_{,m} q^l_{,n} \,, \tag{4.10}$$

and $A^i_{,j}$

$$A^i_{,j} = \frac{1}{2} \epsilon_{jkl} \epsilon^{imn} \frac{\partial q^k}{\partial a^m} \frac{\partial q^l}{\partial a^n} \,, \tag{4.11}$$

where $\epsilon_{ijk} (= \epsilon^{ijk})$ is the Levi-Civita tensor. Using Eq. (4.9) one can show that

$$\frac{\partial \mathcal{J}}{\partial q^i_{,k}} = A^k_{,i} \,, \tag{4.12}$$

which we could also get by diferentiating the cofactor expansion of the determinant. Using the chain rule we get

$$\frac{\partial}{\partial q^k} = \frac{1}{\mathcal{J}} A^i_{,k} \frac{\partial}{\partial a^i} = a^i_{,k} \frac{\partial}{\partial a^i} \,. \tag{4.13}$$

Now, we can combine Eq. (4.12) and Eq. (4.13) to obtain the Euler relation for the time derivative of $\mathcal{J}$:

$$\begin{aligned} \dot{\mathcal{J}} =& \frac{\partial \mathcal{J}}{\partial q^i_{,j}} \dot{q}^i_{,j} = \mathcal{J} a^j_{,i} \dot{q}^i_{,j} \\ =& \mathcal{J} \frac{\partial \dot{q}^i}{\partial q^i} = \mathcal{J} \nabla \cdot u \,. \end{aligned} \tag{4.14}$$

For further discussions see, e.g., Refs. [83, 85, 110].

So far the only property of a fluid element we have considered has been it's position. A fluid element though has more properties. For example, a fluid element can carry a certain density or magnetic flux. These properties are known as the *attributes* of a fluid element and they don't change with time. Therefore, they are only functions of the label and we denote them with the subscript '0'. A fluid element starting out with a density n_0 will carry it forever.

Although the "fluid element", its' position $q(a,t)$ and its' attributes are the fundamental building blocks of the Lagrangian description, the most common description in fluid dynamics is the Eulerian one. It is a field description. By this, we mean that we no longer view the fluid as a collection of fluid elements that we track down their positions and attributes but as a collection of fields that fill a space D. In other words, we no longer ask about e.g. the

velocity of the fluid element a but for the value of the velocity field $v(x,t)$ at position x. The fields of the Eulerian description that characterize the fluid are known as the *observables*.

Since the Lagrangian and the Eulerian descriptions are equivalent, there must be a way to connect the two and translate statements made about the fluid from one description to another. To find this connection we might think in the following way. Imagine that we have a velocity probe with which we can map out the entire velocity field of the fluid. This can be accomplished just by inserting our probe at the position x and measuring the velocity $v(x,t)$ on that point. Yet, by doing this we will have just measured the velocity $\dot{q}(a,t)$ of whatever fluid particle happened to be passing through the point x at the moment we stuck in the probe. A naive way of writing this formally would be $v(x,t) = \dot{q}(a,t)$. However, upon a second look, this relation is clearly incomplete: although we know the position x, since we know exactly the point that we are placing the probe, the above formula doesn't specify which fluid element a will be there. On the other hand, we already know the function $q(a,t)$ and because q is a trajectory, we can learn which fluid element was at the point x, at time t, simply by knowing which elements trajectory q was equal to x at time t. In other words, we are trying to solve $q(a,t) = x$ for a which amounts to inverting the function q. Therefore, the complete statement is:

$$v(x,t) = \dot{q}(a,t)\Big|_{a=q^{-1}(x,t)}. \tag{4.15}$$

Equation 4.15 is known as the Lagrange-to-Euler map and serves as

the "dictionary" for translating statements from one description to the other and vice versa. Of course, it is not the only map of such kind. In principle, we should have maps relating all Eulerian observables to Lagrangian attributes. Of particular importance is the map relating the Lagrangian attribute for mass density ρ_0 with the Eulerian mass density field $\rho(x,t)$. The microscopic statement for mass conservation would be $\rho d^3x = \rho_0 d^3a$. Based on the definition of the Jacobian of the deformation matrix, this amounts to $\rho_0 = \rho\mathcal{J}$ from which we also infer how volume elements from the space of fluid labels translate to volume elements in real space. Again, we need to specify about which fluid element we are talking about therefore, the left hand side needs to be evaluated at $a = q^{-1}(x,t)$. This last step can be written in an integral form if instead of evaluating the Lagrangian attribute function at the inverse function of the trajectory, we use a delta function that selects from all the fluid elements just the right one, i.e the one that passes through x at time t:

$$\rho(x,t) = \int_D d^3a\, \rho_0(a)\delta(x - q(a,t)) = \left.\frac{\rho_0(a)}{\mathcal{J}}\right|_{a=q^{-1}(a,t)} . \tag{4.16}$$

To obtain the second equation, we have evaluated the delta function. Recall that a general property of the delta functions is that:

$$\delta(f(x)) = \sum_{x_i} \frac{\delta(x - x_i)}{|\frac{\partial f}{\partial x}|_{x=x_i}} , \tag{4.17}$$

with x_i being the roots of the equation $f(x) = 0$. In our example, we set $f(a) = x - q(a)$ and the second equation follows immediately.

In the next chapter, we will provide such Euler-Lagrange maps for all the Eulerian observables/ Lagrangian attributes of the theory we will build.

Before we conclude this introductory section on the Eulerian and Lagrangian descriptions, we want to show how they can illuminate the definition of the usual convective derivative used in fluid dynamics. Suppose we start with an Eulerian function of some observable $F = F(x,t)$. As we discussed earlier, the value of F at point x and time t, is the value of the equivalent Lagrangian attribute of the fluid element that happens to be at x at time t. If we take a time derivative of F and view it as a function of q, we find

$$\frac{dF(q(a,t),t)}{dt} = \frac{\partial F}{\partial t} + \dot{q}^i(a,t)\frac{\partial F}{\partial x^i}\bigg|_{a=q^{-1}(x,t)} = \frac{\partial F}{\partial t} + v\cdot\nabla F\,, \tag{4.18}$$

which is the well-known convective derivative.

Chapter 5

Action Principles for Reduced MHD Models

5.1 Introduction

Deriving the equations of motion of fluids [125, 46, 104] and plasmas [76, 118, 13, 26, 97, 105, 55, 85, 84] from an action principle has a rich history. The reasons that such a formulation is pursued, even after the equations of motion are already known, are numerous. Finding conservation laws using Noether's theorem [103, 102, 137, 138], obtaining variational principles for equilibria [2, 3, 92], performing stability analyses [27, 84, 3, 59, 4, 116, 81], or imposing constraints on a theory is straight-forward in the action functional, but often not easily done directly in the equations of motion.

In the study of plasma physics in particular there is one more reason to seek to derive models from a variational principle. As was mentioned in Sec. 1.1 the ad hoc manipulations that we submit the exact moment equations to make them tractable, more often than not, leave us with systems that have been deprived of their Hamiltonian character. Since there is no algorithmic procedure to restore the Hamiltonian structure in an already derived model, a general method that allows us to derive models that are *a priori* Hamiltonian would be of great benefit.

In the present chapter, we develop one such method where we will attempt to derive reduced fluid models directly from an action principle thus guaranteeing that the resulting models will be Hamiltonian (see e.g., Refs. [83, 84, 85, 92]). Such a procedure has the added advantage that the introduction of terms in the action usually has a clear physical meaning. Furthermore, each term of the action functional spawns several terms of the equations of motion. Therefore, when deriving a model performing all orderings, approximations or changes of variables directly in the action, we can immediately understand the exact physical content of the terms that appear or disappear from the equations of motion as a result.

In order for this procedure to work we need to utilize the Lagrangian viewpoint since actions are naturally expressed in terms of Lagrangian variables i.e., functions of the Lagrangian trajectory $q(a,t)$ and its derivatives as well as Lagrangian attributes. Since equations of motion are naturally expressed in Eulerian variables, we will need a Lagrange-to-Euler map that can take the Lagrangian result of the action extremizing to the Eulerian equations of motion. In fact, we go one step further and we only construct actions that can be completely expressed in terms of Eulerian observables. This general requirement was elucidated in Refs. [85, 92], where it was termed the *Eulerian Closure Principle*.

As an example of how this closure requirement for building action principles works, we give the one found in [69]: Consider the kinetic energy term $\frac{1}{2}\int_D d^3a\rho_0(a)\dot{q}^2$. This term can be expressed entirely in terms of the Eulerian

observables of mass density and velocity as $\frac{1}{2}\int_D d^3x\rho v^2$ using the Lagrange-to-Euler maps of Eq. (4.15) and Eq. (4.16). On the other hand, even though the term $\frac{1}{2}\int_D d^3a|B_0(a)|^2$ resembles the magnetic energy, according to Eq. (5.11) it can't be written only in terms of Eulerian observables. Consequently, we don't consider actions that contain such a term as the field energy of the magnetic field.

This chapter is organized as follows: In Sec. 5.2 we review the Lagrangian and Eulerian picture of fluid mechanics, including the derivation of the two-fluid equations of motion through Lagrangian variations of a two-fluid action functional. Starting from this action, we derive a new one-fluid action functional using careful approximations, e.g., imposing quasineutrality, and a change of variables in Sec. 5.3. Here we also introduce a new Lagrange-Euler map and impose locality in order to derive Eulerian equations of motion in the new variables. In Sec. 5.4 we show in detail how to derive various fluid models, e.g., extended MHD and Hall MHD, from this new one-fluid action functional. Sec. 5.5 contains a discussion of Noether's theorem applied to the new action functional. The contents of this chapter are based on the paper of Ref. [60].

5.2 Review: Two-fluid model and action

In this section, we will briefly review the derivation of the non-dissipative two-fluid model equations of motion from the general two-fluid action functional. This action will be the starting point for deriving reduced models

further below. In this context we establish our notation and, later on, discuss differences with our new procedure and results.

As we mentioned in the previous chapter, non-dissipative fluids can be described in two equivalent ways: The Eulerian (or spatial) point of view, which uses the physical observables of, e.g., fluid velocity $v(x,t)$ and mass density $\rho(x,t)$ as its fundamental variables and describes the fluid at an observation point x in the three-dimensional domain as time passes, or the Lagrangian (or material) point of view, which considers individual fluid elements with position q and tracks their time evolution. As described above, both pictures are related through the standard Lagrange-Euler map.

From an action functional/variational point of view, the Lagrangian picture is the more natural one, as it represents the infinite-dimensional generalization of the finite-dimensional Lagrangians of particle mechanics. The equations of motion are then obtained using Hamilton's principle as the stationary points of the action, i.e., the first variation of the action with respect to the variables is equal to zero.

We will use the Lagrangian picture as our starting point and construct a general two-fluid action functional. To ensure physical relevance of the theory, we only construct actions that obey the Eulerian Closure Principle, which states that any action functional of a physical fluid theory must be completely expressible in Eulerian variables after the application of the Lagrange-Euler map.

To simplify our notation (consistent with Ref. [83]), we will avoid explicit vector notation and define the following: $q_s = q_s(a,t)$ is the position of a fluid element ($s = (i,e)$ is the species label) in a rectangular coordinate system where $a = (a_1, a_2, a_3)$ is any label identifying the fluid element and $q_s = (q_{s1}, q_{s2}, q_{s3})$. Here we choose a to be the initial position of the fluid particle at $t = 0$, although other choices are possible [4]. The Lagrangian velocity will then be denoted by $\dot{q}_s$.

The Eulerian velocity field will be denoted by $v_s(x,t)$ with $v_s = (v_{s_1}, v_{s_2}, v_{s_3})$ where $x = (x_1, x_2, x_3)$ is the position in the Eulerian picture. Similarly, we define the electric field vector $E(x,t)$, the magnetic field vector $B(x,t)$, and the vector potential $A(x,t)$. If we need to explicitly refer to components of these vectors, we will use subscripts (or superscripts) j and k. To simplify the equations, we will also often suppress the dependence on x, a, and t.

The action functional described below will include integrations over position space $\int d^3x$ and label space $\int d^3a$. We will not explicitly specify the domains of integration, but assume that our functions are well-defined on their respective domains, and that integrating them and taking functional derivatives is allowed. In addition, we will assume that all variations on the boundaries of the domains and any surface terms (due to integration by parts) vanish.

5.2.1 Constructing the two-fluid action

The action functional of a general theory of a charged fluid interacting with an electromagnetic field should include the following components: The energy of the electromagnetic field, the fluid-field interaction energy, the kinetic energy of the fluid, and the internal energy of the fluid, which describes the fluid's thermodynamic properties.

We will assume two independent fluids corresponding to two different species (ions and electrons with charge e_s, mass m_s and initial number density of $n_{s0}(a)$) which interact with the electromagnetic field, but not directly with each other. Therefore the fluid-dependent parts of the action will naturally split into two parts, one for each species.

The complete action functional is given by

$$S[q_s, A, \phi] = \int_T dt\ L\,, \tag{5.1}$$

where T is a finite time interval and the Lagrangian L is given by

$$L = \frac{1}{8\pi}\int d^3x\ \left[\left|-\frac{1}{c}\frac{\partial A(x,t)}{\partial t} - \nabla\phi(x,t)\right|^2 - |\nabla\times A(x,t)|^2\right] \tag{5.2}$$

$$+\sum_s \int d^3a\ n_{s0}(a)\int d^3x\ \delta\left(x - q_s(a,t)\right)$$

$$\times\left[\frac{e_s}{c}\ \dot{q}_s\cdot A(x,t) - e_s\phi(x,t)\right] \tag{5.3}$$

$$+\sum_s \int d^3a\ n_{s0}(a)\left[\frac{m_s}{2}\ |\dot{q}_s|^2 - m_s U_s\left(m_s n_{s0}(a)/\mathcal{J}_s, s_{s0}\right)\right]. \tag{5.4}$$

Here we have expressed the electric and magnetic fields in terms of the vector and scalar potential, $E = -1/c\,(\partial A/\partial t) - \nabla\phi$ and $B = \nabla\times A$. The first

term (5.2) is the electromagnetic field energy, while the next expression (5.3) is the coupling of the fluid to the electromagnetic field, which is achieved here by using the delta function. The last line of the Lagrangian L (5.4) represents the kinetic and internal energies of the fluid. Note, the specific internal energy (energy per unit mass) of species s, U_s, depends on the Eulerian density as well as a function s_{s0}, an entropy label for each species. Also note, that the vector and scalar potentials are Eulerian variables (i.e., functions of x).

5.2.2 Lagrange-Euler map

In accordance with the above-mentioned Eulerian Closure Principle, we need to ensure that the action Eqs. (5.1)-(5.4) can be completely expressed in terms of the desired set of Eulerian variables, which in turn ensures that the resulting equations of motion will also be completely Eulerian, hence representing a physically meaningful model.

The connection between the Lagrangian and Eulerian pictures of fluids is the *Lagrange-Euler map* which we described in Sec. 4.2. Now, we will implement the ideas found there to obtain relations between all of our Eulerian observables and Lagrangian attributes for this two fluid theory. We define the Eulerian number density $n_s(x,t)$ in terms of Lagrangian quantities as follows:

$$n_s(x,t) = \int d^3a \; n_{s0}(a) \; \delta\left(x - q_s(a,t)\right) \; . \tag{5.5}$$

Using the property of the delta function found in 4.17, this relation can be

rewritten as

$$n_s(x,t) = \left.\frac{n_{s0}(a)}{\mathcal{J}_s}\right|_{a=q_s^{-1}(x,t)}, \tag{5.6}$$

where, $\mathcal{J}_s = \det(\partial q_s/\partial a)$ is the Jacobian determinant. Note that Eq. (5.6) implies the continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla\cdot(n_s v_s) = 0\,, \tag{5.7}$$

which corresponds to local mass conservation if we define the mass density as $\rho_s = m_s n_s$.

The equation for the Eulerian momentum density, $M_s := m_s n_s v_s$, is

$$M_s(x,t) = \int d^3a\, n_{s0}(a,t)\delta\left(x - q_s(a,t)\right) m_s \dot{q}_s(a,t)\,. \tag{5.8}$$

From this one, integrating out the delta function gives us the well-known relation for the Eulerian velocity:

$$v_s(x,t) = \dot{q}_s(a,t)|_{a=q_s^{-1}(x,t)}\,, \tag{5.9}$$

Finally, our Eulerian entropy per unit mass, $s_s(x,t)$, is defined by

$$\rho_s s_s(x,t) = \int d^3a\, n_{s0}(a) s_{s0}(a) m_s\, \delta\left(x - q_s(a,t)\right)\,, \tag{5.10}$$

completing our set of fluid Eulerian variables for this theory, which is $\{n_s, s_s, M_s\}$. It is easy to check that the closure principle is satisfied by these variables.

For the purpose of completeness, had we chosen to write the magnetic field as a Lagrangian attribute of a fluid particle instead of a field quantity, we would have used the following rule to define its components:

$$B^i(x,t) = \int_D d^3a\, q^i_{,j}(a,t) B^j_0(a)\delta(x - q(a,t)) = q^i_{,j}(a,t)\left.\frac{B^j_0(a)}{\mathcal{J}}\right|_{a=q^{-1}(a,t)}. \tag{5.11}$$

5.2.3 Varying the two-fluid action

The action of Eq. (5.1) depends on four dynamical variables: the scalar and vector potentials, ϕ and A, and the positions of the fluid elements q_s.

Varying with respect to ϕ yields Gauss's law

$$\partial_k \left(-\frac{1}{c}\frac{\partial A_k}{\partial t} - \partial_k \phi \right) = 4\pi e \int d^3a \; n_{i0}(a) \; \delta\left(x - q_i\right) - 4\pi e \int d^3a \; n_{e0}(a) \; \delta\left(x - q_e\right) \; ,$$

where $\partial_k := \partial/\partial x^k$, or in more familiar form

$$\nabla \cdot E = 4\pi e \left(n_i(x,t) - n_e(x,t) \right) \; . \tag{5.12}$$

Similarly, the variation with respect to A recovers the Maxwell-Ampere law

$$\frac{1}{4\pi}\left[-\nabla \times \nabla \times A + \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\phi \right) \right] - \frac{e}{c}\int d^3a \; n_{i0} \; \left[\delta\left(x - q_e\right) n_{e0}\dot{q}_e + \delta\left(x - q_i\right)\dot{q}_i \right] = 0$$

or in more familiar form

$$\nabla \times B = \frac{4\pi J}{c} + \frac{1}{c}\frac{\partial E}{\partial t} \, , \tag{5.13}$$

where the Eulerian current density J is defined as

$$J(x,t) = e\left(n_i v_i - n_e v_e \right) \; . \tag{5.14}$$

Recall that the other two Maxwell equations are contained in the definition of the potentials.

Variation with respect to the q_s's is slightly more complex, and we will show a few intermediate steps. Varying the kinetic energy term is straight forward and yields

$$-n_{s0}(a)m_s\ddot{q}_s(a,t) \tag{5.15}$$

The j-th component of the interaction term results in

$$\begin{aligned} & e_s n_{s0}(a)\left[-\frac{1}{c}\frac{dA^j(q_s,t)}{dt}+\frac{1}{c}\dot{q}_s^k\frac{\partial A^k(q_s,t)}{\partial q_s^j}-\frac{\partial\phi(q_s,t)}{\partial q_s^j}\right] \\ & \quad = e_s n_{s0}(a)\left[-\frac{1}{c}\frac{\partial A^j(q_s,t)}{\partial t}-\frac{1}{c}\dot{q}_s^k\frac{\partial A^j(q_s,t)}{\partial q_s^k}\right. \\ & \qquad\qquad \left.+\frac{1}{c}\dot{q}_s^k\frac{\partial A^k(q_s,t)}{\partial q_s^j}-\frac{\partial\phi(q_s,t)}{\partial q_s^j}\right] \\ & \quad = e_s n_{s0}(a)\left[E(q_s,t)+\frac{1}{c}\dot{q}_s(a,t)\times\left(\nabla_{q_s}\times A\left(q_s,t\right)\right)\right]_j . \end{aligned} \tag{5.16}$$

Note that this expression is purely Lagrangian. The fields A and E are evaluated at the positions q_s of the fluid elements and the curl $\nabla_{q_s}\times$ is taken with respect to the Lagrangian position. Also note, since $q_s = q_s(a,t)$, any total time derivative of, e.g., $A(q_s,t)$ will result in two terms. Variation of the internal energy term yields

$$-A_i^j\frac{\partial}{\partial a_j}\left(\frac{\rho_{s0}^2}{\mathcal{J}_s^2}\frac{\partial U\left(\frac{\rho_{s0}}{\mathcal{J}_s},s_{s0}\right)}{\partial\left(\frac{\rho_{s0}}{\mathcal{J}_s}\right)}\right) . \tag{5.17}$$

Setting the sum of Eqs. (5.15)-(5.17) equal to zero and invoking the usual thermodynamic relations between internal energy and pressure and temperature,

$$p_s = (m_s n_s)^2\frac{\partial U_s}{\partial(m_s n_s)} \quad \text{and} \quad T_s = \frac{\partial U_s}{\partial s_s}\,, \tag{5.18}$$

results in the well-known (non-dissipative) two-fluid equations of motion

$$m_s n_s \left(\frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s \right) = e_s n_s \left(E + \frac{1}{c} v_s \times B \right) - \nabla p_s \tag{5.19}$$

Further analysis (see e.g. Refs. [77, 29]) of these equations usually involves the addition and subtraction of the two-fluid equations and a change of variable transformation to

$$\begin{aligned} V &= \frac{1}{\rho_m} \left(m_i n_i v_i + m_e n_e v_e \right) \\ J &= e \left(n_i v_i - n_e v_e \right) \\ \rho_m &= m_i n_i + m_e n_e \\ \rho_q &= e \left(n_i - n_e \right) . \end{aligned} \tag{5.20}$$

The resulting equations are then simplified by, e.g., making certain assumptions (quasineutrality, $v << c$, etc.) and ordering to obtain two new *one*-fluid equations – one often referred to as the *one-fluid momentum equation* and the other as the *generalized Ohm's law*.

5.3 The new one-fluid action

The first step in building an action functional for fluid models is to decide on the relevant Eulerian observables of the model. Since we want to derive, e.g., the two-fluid model of Lüst and various reductions, our Eulerian observables are going to be the set $\{n, s, s_e, V, J, E, B\}$, where $s = (m_i s_i + s_e m_e)/m$, with $m = m_e + m_i$, and n is a single number density variable.

Next we have to define our Lagrangian variables and with them construct the action. Any additional assumption (e.g., quasineutrality etc.) and ordering will be implemented on the action level. Varying the new action will then result in equations of motion that, using properly defined Lagrange-Euler maps, will Eulerianize to, e.g., Lüst's equation of motion and the generalized Ohm's law.

5.3.1 New Lagrangian variables

We will start by defining a new set of Lagrangian variables, inspired by Eq. (5.20)[1],

$$\begin{aligned} Q(a,t) &= \frac{1}{\rho_{m0}(a)}\left(m_i n_{i0}(a) q_i(a,t) + m_e n_{e0}(a) q_e(a,t)\right) \\ D(a,t) &= e\left(n_{i0}(a) q_i(a,t) - n_{e0}(a) q_e(a,t)\right) \\ \rho_{m0}(a) &= m_i n_{i0}(a) + m_e n_{e0}(a) \\ \rho_{q0}(a) &= e\left(n_{i0}(a) - n_{e0}(a)\right) . \end{aligned} \tag{5.21}$$

Here $Q(a,t)$ can be interpreted as a center of mass position variable and $D(a,t)$ as a local dipole moment variable, connecting an ion fluid element to an electron fluid element. It is then straight-forward to take the time-derivative of Q and D which can be interpreted as the center-of-mass velocity $\dot{Q}(a,t)$ and a Lagrangian current $\dot{D}(a,t)$, respectively. Using appropriately defined

[1]First presented in A. Wurm and P.J. Morrison, Action principle derivation of one-fluid models from two-fluid actions, Bulletin of the Am. Phys. Soc, Vol. 54, Nr. 4 (2009).

Lagrange-Euler maps, $\dot{Q}(a,t)$ will map to the Eulerian velocity $V(x,t)$ and $\dot{D}(a,t)$ to the Eulerian current $J(x,t)$ as defined by Eq. (5.20).

We will also need the inverse of this transformation,

$$\begin{aligned} q_i(a,t) &= \frac{\rho_{m0}(a)Q(a,t) + \frac{m_e}{e}D(a,t)}{\rho_{m0}(a) + \frac{m_e}{e}\rho_{q0}} \\ q_e(a,t) &= \frac{\rho_{m0}(a)Q(a,t) - \frac{m_i}{e}D(a,t)}{\rho_{m0}(a) - \frac{m_i}{e}\rho_{q0}} \\ n_{i0}(a) &= \frac{\rho_{m0}(a) + \frac{m_e}{e}\rho_{q0}(a)}{m} \\ n_{e0}(a) &= \frac{\rho_{m0}(a) - \frac{m_e}{e}\rho_{q0}(a)}{m}\,. \end{aligned} \tag{5.22}$$

5.3.2 Ordering of fields and quasineutrality

Typically, reductions of the full two-fluid model are obtained by imposing an auxiliary ordering on the equations of motion. In order to preserve the variational formulation, we perform an ordering directly in the action.

To construct the action, we will start with the two-fluid action of Eq. (5.1) and change variables to Q and D, but in light of the fluid models we are interested in, we will first make two simplifying assumptions: We order the fields in the action so that the displacement current in Eq. (5.13) will vanish, and we assume quasineutrality. In this section, we describe this field ordering in detail and discuss quasineutrality in the Lagrangian variable context, which as far as we know has not been done before.

The omission of the displacement current is allowed, when the time

scale of changes in the field configuration is long relative to the time it takes for radiation to "communicate" these changes across the system [106]. We use non-dimensional variables by introducing a characteristic scale B_0 for the magnetic field and a characteristic length scale ℓ for gradients. Times are then normalized by the Alfvén time $t_A = B_0/\sqrt{4\pi\rho}$ and the $\dot{q}_s$'s by the Alfvén speed $v_A = \ell/t_A$, resulting in the following form for the sum of the field and interaction terms of the Lagrangian (5.4):

$$\frac{B_0^2}{8\pi}\int dt \int d^3\hat{x}\left[\left|-\frac{v_A}{c}\frac{\partial \hat{A}}{\partial \hat{t}} - \frac{\phi_0}{B_0\ell}\hat{\nabla}\hat{\phi}\right|^2 - \left|\hat{\nabla}\times\hat{A}\right|^2\right]$$
$$+\sum_s B_0^2\left[\int dt\int d^3\hat{a}\; n_0\hat{n}_{s0}(a)e_s\int d^3\hat{x}\;\delta\left(\hat{x}-\hat{q}_s\right)\times\left(\frac{v_A\ell}{B_0 c}\hat{\dot{q}}_s\cdot\hat{A} - \frac{\phi_o}{B_0^2}\hat{\phi}\right)\right],$$

where ϕ_0 and n_0 are yet to be specified scales for the electrostatic potential and the densities of both species, respectively. We also require that the two species' velocities are of the same scale. Requiring the two interaction terms in the Lagrangian to be of the same order results in a scaling for ϕ; viz., $\phi_0 \equiv B_0\ell v_A/c$. Thus, both parts of the $|E|^2$ term are of order $\mathcal{O}\left(v_A/c\right)$. Neglecting this term and varying with respect to $\hat{A}$ results in

$$\hat{\nabla}\times\hat{B} = \frac{4\pi e n_0 v_A}{c}\frac{\ell}{B_0}\left(\int d^3a\;\delta(\hat{x}-\hat{q}_i)\hat{n}_{i0}(a)\hat{\dot{q}}_i - \int d^3a\;\delta(\hat{x}-\hat{q}_e)\hat{n}_{e0}(a)\hat{\dot{q}}_e\right),$$

which can be written as

$$\frac{B_0}{\ell}\hat{\nabla}\times\hat{B} = \frac{4\pi j_0}{c}\hat{J}\,, \tag{5.23}$$

where $j_0 = e n_0 v_A$ is a scale for the current.

Varying the scaled action with respect to $\hat{\phi}$ yields

$$0 = \int d^3\hat{a}\;\delta(\hat{x}-\hat{q}_i)\hat{n}_i - \int d^3\hat{a}\;\delta(\hat{x}-\hat{q}_e)\hat{n}_e \equiv \Delta\hat{n}\,. \tag{5.24}$$

The above equation states that the difference in the two densities is zero, i.e., the plasma is quasineutral, a property that holds locally, i.e., $n_i(x,t) = n_e(x,t)$. Using Eq. (5.6), this statement would correspond to the following in the Lagrangian variable picture:

$$\left.\frac{n_{i0}(a)}{\mathcal{J}_i(a,t)}\right|_{a=q_i^{-1}(x,t)} = \left.\frac{n_{e0}(a)}{\mathcal{J}_e(a,t)}\right|_{a=q_e^{-1}(x,t)} . \tag{5.25}$$

In the Lagrangian picture we will make the additional assumption of homogeneity: $n_{i0}(a) = n_{e0}(a) =$ constant, which is natural for the plasma we are modeling. It states that at $t = 0$ all fluid elements are identical in the amount of density they carry. Therefore, n_{i0} and n_{e0} can be replaced by a constant n_0. Equation (5.25) then reduces to a statement about the two Jacobians

$$\left.\mathcal{J}_i(a,t)\right|_{a=q_i^{-1}(x,t)} = \left.\mathcal{J}_e(a,t)\right|_{a=q_e^{-1}(x,t)} , \tag{5.26}$$

which will play a central role in our development below.

Note, that the homogeneity assumption ($n_{i0} = n_{e0} = n_0$) does not prohibit us from describing quasineutral plasmas with density gradients. What we would have to do in this case, would be to pick our labeling scheme, and hence the Jacobian, accordingly, as to reflect the initial density gradient of the configuration. Thus there is freedom in this regard beyond what we are assuming now.

5.3.3 Action functional

We are now ready to implement the change of variables discussed in Sec. 5.3.1. Because of the homogeneity assumption $n_{i0}(a) = n_{e0}(a) = n_0$, the new variables of Eq. (5.21) reduce to

$$\begin{aligned} Q(a,t) &= \frac{m_i}{m} q_i(a,t) + \frac{m_e}{m} q_e(a,t) \\ D(a,t) &= e n_0 \left(q_i(a,t) - q_e(a,t) \right) \\ \rho_{m0}(a) &= m n_0 \\ \rho_{q0}(a) &= 0 \end{aligned} \tag{5.27}$$

and the inverse transformation of Eq. (5.22) to

$$\begin{aligned} q_i(Q,D) := q_i(a,t) &= Q(a,t) + \frac{m_e}{m e n_0} D(a,t) \\ q_e(Q,D) := q_e(a,t) &= Q(a,t) - \frac{m_i}{m e n_0} D(a,t) \,, \end{aligned} \tag{5.28}$$

where we choose the notation $q_s(Q,D)$ to emphasize that the q_s should not be thought of as ion/electron trajectories any more but as specific linear combinations of $Q(a,t)$ and $D(a,t)$. In addition, we will need the ion and electron Jacobians, $\mathcal{J}_i(Q,D)$ and $\mathcal{J}_e(Q,D)$, now expressed in terms of Q and D.

The resulting action functional has the form:

$$
\begin{aligned}
S = &- \frac{1}{8\pi} \int dt \int d^3x \ |\nabla \times A(x,t)|^2 \\
&+ \int dt \int d^3x \int d^3a \ n_0 \left\{ \delta(x - q_i(Q,D)) \left[\frac{e}{c} \left(\dot{Q}(a,t) + \frac{m_e}{men_0} \dot{D}(a,t) \right) \cdot A(x,t) - e\phi(x,t) \right] \right\} \\
&+ \int dt \int d^3x \int d^3a \ n_0 \left\{ \delta(x - q_e(Q,D)) \left[-\frac{e}{c} \left(\dot{Q}(a,t) - \frac{m_i}{men_0} \dot{D}(a,t) \right) \cdot A(x,t) + e\phi(x,t) \right] \right\} \\
&\quad + \frac{1}{2} \int dt \int d^3a \ n_0 \left[m_i |\dot{Q}|^2(a,t) + \frac{m_i m_e}{me^2 n_0^2} |\dot{D}|^2(a,t) \right] \\
&\quad - \int dt \int d^3a \ n_0 \left[m_i U_i \left(\frac{m_i n_0}{\mathcal{J}_i(Q,D)}, (ms_0 - m_e s_{e0})/m_i \right) + m_e U_e \left(\frac{m_e n_0}{\mathcal{J}_e(Q,D)}, s_{e0} \right) \right] ,
\end{aligned}
\tag{5.29}
$$

where recall $s_0 = (m_i s_{i0} + m_e s_{e0})/m$.

5.3.4 Nonlocal Lagrange-Euler maps

Now we define the Lagrange-Euler maps that connect the Eulerian observables V and J to the new Lagrangian variables Q and D. Referring to Sec. 4.2, one can see that a Lagrange-Euler map is a relationship between a Lagrangian quantity and some Eulerian observables, which holds only when it is evaluated on a trajectory $x = q_s(a,t)$. If we apply the inverse Lagrange-Euler maps from Eqs. (5.6) and (5.9) to Eq. (5.20) and assume quasineutrality,

we get

$$
\begin{aligned}
V(x,t) &= \frac{m_i}{m}\,\dot{q}_i(a,t)\Big|_{a=q_i^{-1}(x,t)} + \frac{m_e}{m}\,\dot{q}_e(a,t)\Big|_{a=q_e^{-1}(x,t)} \\
J(x,t) &= e\left(\frac{n_0}{\mathcal{J}_i(a,t)}\dot{q}_i(a,t)\right)\Big|_{a=q_i^{-1}(x,t)} - e\left(\frac{n_0}{\mathcal{J}_e(a,t)}\dot{q}_e(a,t)\right)\Big|_{a=q_e^{-1}(x,t)} \\
n(x,t) &= \frac{m_i}{m}\left(\frac{n_0}{\mathcal{J}_i(a,t)}\right)\Big|_{a=q_i^{-1}(x,t)} + \frac{m_e}{m}\left(\frac{n_0}{\mathcal{J}_e(a,t)}\right)\Big|_{a=q_e^{-1}(x,t)} \qquad (5.30) \\
s(x,t) &= \frac{m_i}{m}\,s_{i0}\Big|_{a=q_i^{-1}(x,t)} + \frac{m_e}{m}\,s_{e0}\Big|_{a=q_e^{-1}(x,t)} \qquad (5.31) \\
s_e(x,t) &= s_{e0}\Big|_{a=q_e^{-1}(x,t)}\,. \qquad (5.32)
\end{aligned}
$$

The definitions of $Q(a,t)$ and $D(a,t)$ in Eq. (5.27) suggest that their time-derivatives should be associated with V and J, respectively. However, both $\dot{Q}$ and $\dot{D}$ are nonlocal objects, since they relate the velocities of electrons and ions which are located at different points in space. This means that neither $\dot{Q}$, nor $\dot{D}$, when evaluated at the inverse maps for a, can Eulerianize to a local velocity or current, since, in general, $x = q_i(Q,D)$ and $x' = q_e(Q,D)$ with $x \neq x'$ or, they are simultaneously evaluated at different trajectories. Therefore, we have two different inverse functions where the Lagrangian quantities are to be evaluated, namely, $a = q_i^{-1}(x,t)$ and $a = q_e^{-1}(x',t)$ which should be thought of as the inverse functions of $x = q_i(Q,D)$ and $x' = q_e(Q,D)$. To make this

work, we *define* our Lagrange-Euler maps with $x = x'$ as

$$\begin{aligned} V(x,t) = & \frac{m_i}{m}\left(\dot{Q}(a,t) + \frac{m_e}{men_0}\dot{D}(a,t)\right)\Big|_{a=q_i^{-1}(x,t)} \\ & + \frac{m_e}{m}\left(\dot{Q}(a,t) - \frac{m_i}{men_0}\dot{D}(a,t)\right)\Big|_{a=q_e^{-1}(x,t)} \\ J(x,t) = & \frac{en_0}{\mathcal{J}_i(a,t)}\left(\dot{Q}(a,t) + \frac{m_e}{men_0}\dot{D}(a,t)\right)\Big|_{a=q_i^{-1}(x,t)} \\ & - \frac{en_0}{\mathcal{J}_e(a,t)}\left(\dot{Q}(a,t) - \frac{m_i}{men_0}\dot{D}(a,t)\right)\Big|_{a=q_e^{-1}(x,t)} . \end{aligned} \tag{5.33}$$

Due to Eq. (5.26), the two Jacobian determinants are equal (as long as they are evaluated at the respective inverse functions) and can be replaced by a common Jacobian determinant, $\mathcal{J}$.

The maps just defined are straight-forward to apply for mapping an Eulerian statement to a Lagrangian one, but for our purpose, we have to invert them. To keep careful track of the two inverse functions, we first invert the intermediate relations

$$\begin{aligned} & V(x,t) + \frac{m_e}{men(x,t)}J(x,t) \\ & \qquad = \left(\dot{Q}(a,t) + \frac{m_e}{men_0}\dot{D}(a,t)\right)\Big|_{a=q_i^{-1}(x,t)} \end{aligned} \tag{5.34}$$

$$\begin{aligned} & V(x,t) - \frac{m_i}{men(x,t)}J(x,t) \\ & \qquad = \left(\dot{Q}(a,t) - \frac{m_i}{men_0}\dot{D}(a,t)\right)\Big|_{a=q_e^{-1}(x,t)} , \end{aligned} \tag{5.35}$$

where we have used Eq. (5.6). The inverse Lagrange-Euler maps are now given

by

$$\begin{aligned}
\dot{Q}(a,t) &= \frac{m_i}{m}\left(V(x,t)+\frac{m_e}{men(x,t)}J(x,t)\right)\Bigg|_{x=q_i(Q,D)} \\
&+ \frac{m_e}{m}\left(V(x',t)-\frac{m_i}{men(x',t)}J(x',t)\right)\Bigg|_{x'=q_e(Q,D)} \\
\dot{D}(a,t) &= en_0\left(V(x,t)+\frac{m_e}{men(x,t)}J(x,t)\right)\Bigg|_{x=q_i(Q,D)} \\
&- en_0\left(V(x',t)-\frac{m_i}{men(x',t)}J(x',t)\right)\Bigg|_{x'=q_e(Q,D)} .
\end{aligned} \tag{5.36}$$

Note that the construction of the maps of Eqs. (5.33) and (5.36) can be done with any invertible linear combination of the time derivatives of our Lagrangian variables. The only restriction is that the action should comply with the Eulerian Closure Principle, i.e., it should be expressible entirely in terms of the Eulerian observables. It is straightforward to show that this is true in our case.

5.3.5 Lagrange-Euler maps without quasineutrality

Had we not assumed quasineutrality, we would have to proceed differently: Eq. (5.20) implies that the proper Lagrangian variables that would

Eulerianize to velocity and current would be

$$V(x,t) = \frac{m_i \left(\frac{n_{i0}}{\mathcal{J}_i}\dot{q}_i(a,t)\right)\Big|_{a=q_i^{-1}(x,t)}}{m_i \left(\frac{n_{i0}}{\mathcal{J}_i}\right)\Big|_{a=q_i^{-1}(x,t)} + m_e \left(\frac{n_{e0}}{\mathcal{J}_e}\right)\Big|_{a=q_e^{-1}(x,t)}} + \frac{m_e \left(\frac{n_{e0}}{\mathcal{J}_e}\dot{q}_e(a,t)\right)\Big|_{a=q_e^{-1}(x,t)}}{m_i \left(\frac{n_{i0}}{\mathcal{J}_i}\right)\Big|_{a=q_i^{-1}(x,t)} + m_e \left(\frac{n_{e0}}{\mathcal{J}_e}\right)\Big|_{a=q_e^{-1}(x,t)}},$$

$$J(x,t) = e\left(\frac{n_{i0}}{\mathcal{J}_i}\dot{q}_i(a,t)\right)\Big|_{a=q_i^{-1}(x,t)} - e\left(\frac{n_{e0}}{\mathcal{J}_e}\dot{q}(a,t)\right)\Big|_{a=q_e^{-1}(x,t)}.$$

The above equations suggest that without quasineutrality, the definitions for $\dot{Q}$, $\dot{D}$, etc. should be modified to the following:

$$\dot{Q}(a,t) = \frac{1}{\rho_{m0}(a)}\big(m_i\mathcal{J}_e n_{i0}(a)\dot{q}_i(a,t) + m_e\mathcal{J}_i n_{e0}(a)\dot{q}_e(a,t)\big)$$
$$\dot{D}(a,t) = e\big(\mathcal{J}_e n_{i0}(a)\dot{q}_i(a,t) - \mathcal{J}_i n_{e0}(a)\dot{q}_e(a,t)\big)$$
$$\rho_{m0}(a) = m_i\mathcal{J}_e n_{i0}(a) + m_e\mathcal{J}_i n_{e0}(a)$$

where $\dot{Q}/\left(\mathcal{J}_i\mathcal{J}_e\right)$ maps to $V(x,t)$ and $\dot{D}/\left(\mathcal{J}_i\mathcal{J}_e\right)$ to $J(x,t)$. In this case, however, both $\dot{Q}$ and $\dot{D}$ are implicitly defined, since $\mathcal{J}_i$ and $\mathcal{J}_e$ depend on them. This problem is absent when only manipulating the Eulerian equations of motion. It might suggest though that when quasineutrality does not hold, the one-fluid description might not be appropriate. To see why this is so, consider the following example: in order to satisfy the ECP for the kinetic energy term, we swap the label volume element with the position volume element, divided by a Jacobian. This Jacobian combines with the n_0 to give the Eulerian density $n(x,t)$. When quasineutrality does not hold, we would have instead an n_{i0}

and an n_{e0}, each needing each own Jacobian to be Eulerianized. This means, that within the one-fluid formalism, we would still be required to keep track of electron and ion labels, i.e., have two different label space volume elements. This can also be seen in the most general case derived by Lüst in Ref. [77]. The resulting equations of motion in V and J still contain terms explicitly referring to ion/electron quantities, e.g., n_i and n_e. In conclusion, from a variational point of view, it is not obvious how to apply the Eulerian Closure Principle without quasineutrality, within the context of a single fluid model. It seems that in order to preserve it, one would need to distinguish between integrations over ion and electron labels, so that the d^3a could be related to the proper $\mathcal{J}_s$.

5.3.6 Derivation of the continuity and entropy equations

Before we derive the equations of motion for several different models in the next section, we derive here the continuity equation, which all of the models below have in common, and the entropy equations.

Due to the identity of the Jacobians from Eq. (5.26), Eq. (5.30) for n reduces to

$$n(x,t) = \left(\frac{n_0}{\mathcal{J}_i(a,t)}\right)\Bigg|_{a=q_i^{-1}(x,t)} = \left(\frac{n_0}{\mathcal{J}_e(a,t)}\right)\Bigg|_{a=q_e^{-1}(x,t)}, \tag{5.37}$$

where q_s^{-1} are still the inverse functions of $q_s(Q,D)$. Inverting the equation for the ions and taking the time derivative yields

$$\left.\frac{dn}{dt}\right|_{x=q_i(Q,D)} = \frac{d}{dt}\frac{n_0}{\mathcal{J}_i(a,t)} = -\frac{n_0}{\mathcal{J}_i^2(a,t)}\frac{\partial \mathcal{J}_i}{\partial t}\,.$$

To Eulerianize the equation above, we use the well-known relations $d/dt = \partial/\partial t + v \cdot \nabla$ and $\partial \mathcal{J}/\partial t = \mathcal{J} \nabla \cdot v$ that we proved in Sec. 4.2. The key here is to use the correct Eulerian velocity, in this case the ion velocity in terms of V and J. The result is

$$\frac{\partial n}{\partial t} + \left(V + \frac{m_e}{men} J \right) \cdot \nabla n = -n \nabla \cdot \left(V + \frac{m_e}{men} J \right)$$

which can be further reduced to

$$\frac{\partial n}{\partial t} + \nabla \cdot (nV) + \frac{m_e}{me} \nabla \cdot J = 0 \,.$$

However, we already know from Eq. (5.23) that $\nabla \cdot J = 0$. Therefore, no matter which equality we choose in Eq. (5.37), the continuity equation will be the same,

$$\frac{\partial n}{\partial t} + \nabla \cdot (nV) = 0 \,. \tag{5.38}$$

Similarly, from Eq. (5.31) we obtain

$$\frac{\partial s}{\partial t} + V \cdot \nabla s = 0$$

and from Eq. (5.32)

$$\frac{\partial s_e}{\partial t} + \left(V - \frac{m_i}{men} J \right) \cdot \nabla s_e = 0 \,,$$

or to leading order in m_e/m_i

$$\frac{\partial s_e}{\partial t} + \left(V - \frac{1}{en} J \right) \cdot \nabla s_e = 0 \,.$$

5.4 Derivation of reduced models

If we vary the action functional (5.29) with respect to Q and D and subsequently apply the Lagrange-Euler map we recover the momentum equation and generalized Ohm's law of Lüst[2] (in the non-dissipative limit):

$$nm\left(\frac{\partial V}{\partial t}+(V\cdot\nabla)V\right) \tag{5.39}$$

$$\begin{aligned} &= -\nabla p+\frac{J\times B}{c}-\frac{m_i m_e}{me^2}(J\cdot\nabla)\left(\frac{J}{n}\right) \\ E+\frac{V\times B}{c} &= \frac{m_i m_e}{me^2 n}\left(\frac{\partial J}{\partial t}+(J\cdot\nabla)V-(J\cdot\nabla)\left(\frac{J}{n}\right)+(\nabla\cdot V)J\right) \\ &\quad +\frac{m_i m_e}{me^2}(V\cdot\nabla)\left(\frac{J}{n}\right)+\frac{(m_i-m_e)}{menc}(J\times B) \\ &\quad -\frac{m_i}{men}\nabla p_e+\frac{m_e}{men}\nabla p_i\,. \end{aligned} \tag{5.40}$$

We will not show this lengthy, although straightforward, calculation here, but instead show the detailed derivation of extended MHD in the next section which requires one more ordering in the action of Eq. (5.29).

5.4.1 Extended MHD

At this point we will make one more simplification: We define the mass ratio $\mu = m_e/m_i$ and order the action functional keeping terms up to first

[2]Note, there are typos in Eqs. (2.9) and (2.10) of Ref. [77] that prevent the term N_1 from vanishing when imposing quasineutrality, as it should.

order in μ. Up to first order, the change of variables is

$$\begin{aligned} q_i(Q,D) &= Q(a,t) + \frac{\mu}{en_0} D(a,t) \\ q_e(Q,D) &= Q(a,t) - \frac{1-\mu}{en_0} D(a,t) \end{aligned} \tag{5.41}$$

and the action takes on the form

$$\begin{aligned} S = &-\frac{1}{8\pi}\int dt \int d^3x \ |\nabla \times A(x,t)|^2 \\ &+ \int dt \int d^3x \int d^3a \ n_0 \Bigg\{ \delta(x - q_i(Q,D)) \\ &\quad \times \left[\frac{e}{c}\dot{Q}(a,t) + \frac{\mu}{cn_0}\dot{D}(a,t)\cdot A(x,t) - e\phi(x,t) \right] \Bigg\} \\ &+ \int dt \int d^3x \int d^3a \ n_0 \Bigg\{ \delta(x - q_e(Q,D)) \\ &\quad \times \left[-\frac{e}{c}\dot{Q}(a,t) + \frac{(1-\mu)}{cn_0}\dot{D}(a,t)\cdot A(x,t) + e\phi(x,t) \right] \Bigg\} \\ &+ \frac{1}{2}\int dt \int d^3a \, n_0 m_i \left((1+\mu)|\dot{Q}|^2(a,t) + \frac{\mu}{e^2 n_0^2}|\dot{D}|^2(a,t) \right) \\ &- \int dt \int d^3a \, n_0 \Bigg(\mathfrak{U}_e \left(\frac{n_0}{\mathcal{J}_e(Q,D)}, s_{e0} \right) \\ &\qquad\qquad + \mathfrak{U}_i \left(\frac{n_0}{\mathcal{J}_i(Q,D)}, s_{i0} \right) \Bigg) . \end{aligned} \tag{5.42}$$

where for convenience we have replaced the U_s, the internal energy per unit mass, by $\mathfrak{U}_s$, the internal energy per particle. The pressure is obtained from the latter according to $p_s = n^2 \partial \mathfrak{U}_s / \partial n$.

Varying the action with respect to Q_k yields

$$
\begin{aligned}
0 = & -n_0 m_i (1+\mu)\ddot{Q}_k(a,t) - \partial_k p \\
& + n_0 \left[\frac{e}{c} \left(\dot{Q}_j(a,t) + \frac{\mu}{e n_0} \dot{D}_j(a,t) \right) \frac{\partial A_j(x,t)}{\partial x^k} - e\partial_k \phi(x,t) - \frac{e}{c} \frac{d}{dt} A_k(x,t) \right] \Bigg|_{x=q_i(Q,D)} \\
& + n_0 \left[-\frac{e}{c} \left(\dot{Q}_j(a,t) - \frac{(1-\mu)}{e n_0} \dot{D}_j(a,t) \right) \frac{\partial A_j(x,t)}{\partial x^k} + e\partial_k \phi(x,t) + \frac{e}{c} \frac{d}{dt} A_k(x,t) \right] \Bigg|_{x=q_e(Q,D)} .
\end{aligned}
\tag{5.43}
$$

The variation of the internal energy term proceeds by varying q_s through Eqs. (5.41), giving $\delta q_s = \delta Q$ and using these expressions in the variation of the Jacobians $\mathcal{J}_s$. We have given the Eulerian result since the Lagrangian one has two terms of the form of Eq. (5.17), and it is cumbersome to carry this through the rest of the calculation. (See Ref. [63] for a treatment that orders the Eulerian equations directly.) Consistent with Dalton's law, the total single fluid pressure is $p = p_i + p_e$ and both these pressures come in entirely at the zeroth order of μ. Note, that the two time derivatives of A do not cancel, because they are advected by different flows, or, since we are still in the Lagrangian framework, they are evaluated at different arguments.

To find the Eulerian equations of motion, we start with Eq. (5.36) (up to first order in μ) and impose locality, i.e., $x = x'$, such that $\dot{Q}$ maps to $V(x,t)$ and $\dot{D}$ to $J(x,t)$. However, the time derivatives of $\dot{Q}$ and $\dot{D}$ have to be treated with care as they each consist of two terms that are advected with different velocities. We will show how to Eulerianize the equations of motion in detail.

The $\ddot{Q}$ in the first term of Eq. (5.43) can be re-written as

$$\begin{aligned}
&\ddot{Q}(a,t) \qquad (5.44)\\
&= (1-\mu)\,\frac{d}{dt}\left(V(x,t)+\frac{\mu}{en(x)}J(x,t)\right)\Big|_{x=q_i(Q,D)}\\
&\quad + \mu\frac{d}{dt}\left(V(x,t)-\frac{(1-\mu)}{en(x)}J(x,t)\right)\Big|_{x=q_e(Q,D)}\\
&= (1-\mu)\left(\frac{\partial V}{\partial t}+\frac{\partial q_i}{\partial t}\cdot\nabla V+\frac{\mu}{en}\frac{\partial}{\partial t}\left(\frac{J}{n}\right)\right. \qquad (5.45)\\
&\quad \left.+\frac{\mu}{e}\frac{\partial q_i}{\partial t}\cdot\nabla\left(\frac{J}{n}\right)\right)+\mu\left(\frac{\partial V}{\partial t}+\frac{\partial q_e}{\partial t}\cdot\nabla V\right.\\
&\quad \left.-\frac{(1-\mu)}{en}\frac{\partial}{\partial t}\left(\frac{J}{n}\right)-\frac{(1-\mu)}{e}\frac{\partial q_e}{\partial t}\cdot\nabla\left(\frac{J}{n}\right)\right)\,.
\end{aligned}$$

From Eqs. (5.41), we can find explicit expressions for the time derivatives of the $q_s(Q,D)$,

$$\frac{\partial q_i}{\partial t}=\dot{Q}+\frac{\mu}{en_0}\dot{D}\longrightarrow V+\frac{\mu}{en}J \qquad (5.46)$$

$$\frac{\partial q_e}{\partial t}=\dot{Q}-\frac{1-\mu}{en_0}\dot{D}\longrightarrow V-\frac{1-\mu}{en}J\,. \qquad (5.47)$$

Inserting these expression into Eq. (5.45), we find after some algebra that

$$\ddot{Q}(a,t)\longrightarrow\frac{\partial V}{\partial t}+(V\cdot\nabla)V+\frac{\mu(1-\mu)}{ne^2}(J\cdot\nabla)\left(\frac{J}{n}\right)\,. \qquad (5.48)$$

Next we Eulerianize the interaction terms of Eq. (5.43) using Eq. (5.36) (up to first order in μ) and Eq. (5.41). The result is

$$\begin{aligned}
&\frac{ne}{c}\left[\left(V_j+\frac{\mu}{en}J_j\right)\frac{\partial A_j}{\partial x^k}-c\partial_k\phi-\frac{\partial A_k}{\partial t}-\frac{\partial q_i}{\partial t}\cdot\nabla A_k\right]\\
&+\frac{ne}{c}\left[\left(-V_j+\frac{(1-\mu)}{en}J_j\right)\frac{\partial A_j}{\partial x^k}+c\partial_k\phi+\frac{\partial A_k}{\partial t}+\frac{\partial q_e}{\partial t}\cdot\nabla A_k\right]\,, \qquad (5.49)
\end{aligned}$$

which, after substitution using Eqs. (5.46) and (5.47), yields

$$\frac{1}{c}\left(J_j\frac{\partial A_j}{\partial x^k}-J_j\frac{\partial A_k}{\partial x^j}\right)=\frac{(J\times(\nabla\times A))_k}{c}\,. \tag{5.50}$$

The full Eulerian version of the equation of motion for the velocity of Eq. (5.43), also referred to as the *momentum equation* is

$$nm\left(\frac{\partial V}{\partial t}+(V\cdot\nabla)V\right)=-\nabla p+\frac{J\times B}{c}-\frac{m_e}{e^2}(J\cdot\nabla)\left(\frac{J}{n}\right)\,. \tag{5.51}$$

Note, it was shown in Ref. [63] that the last term of Eq. (5.51) is necessary for energy conservation.

Next, varying the action with respect to D_k yields

$$\begin{aligned}0=&\frac{m_i\mu}{n_0e^2}\ddot{D}_k(a,t)+\frac{(1-\mu)}{en_0}\partial_k p_e-\frac{\mu}{en_0}\partial_k p_i\\&+\mu\left[\left(-\frac{1}{c}\frac{d}{dt}A_k(x,t)-\partial_k\phi(x,t)+\frac{1}{c}\left(\dot{Q}_j(a,t)\quad+\frac{\mu}{en_0}\dot{D}_j(a,t)\right)\frac{\partial A_j(x,t)}{\partial x^k}\right)\right]\Bigg|_{x=q_i(Q,D)}\\&+(1-\mu)\left[\left(-\frac{1}{c}\frac{d}{dt}A_k(x,t)-\partial_k\phi(x,t)+\frac{1}{c}\left(\dot{Q}_j(a,t)\quad-\frac{(1-\mu)}{en_0}\dot{D}_j(a,t)\right)\frac{\partial A_j(x,t)}{\partial x^k}\right)\right]\Bigg|_{x=q_e(Q,D)}\end{aligned} \tag{5.52}$$

This time the Jacobians of the internal energies are varied using $\delta q_e=-(1-\mu)\delta D/(en_0)$ and $\delta q_i=\mu\delta D/(en_0)$, which again follow from Eqs. (5.41). Note, it is for this reason that only the electron pressure appears to leading order in Ohm's law for extended MHD.

Eulerianizing the $\ddot{D}$ term in Eq. (5.52) yields

$$\begin{aligned}\frac{m_i\mu}{n_0e^2}\ddot{D}(a,t)=&\frac{m_i\mu}{e^2n}\left(\frac{\partial J}{\partial t}+(J\cdot\nabla)V-(J\cdot\nabla)\left(\frac{J}{n}\right)+(\nabla\cdot V)J\right)\\&+\frac{m_i\mu}{e^2}(V\cdot\nabla)\left(\frac{J}{n}\right)+\frac{m_i\mu}{e^2n^2}J(V\cdot\nabla)n\,,\end{aligned} \tag{5.53}$$

where we have used the continuity equation Eq. (5.38) to eliminate the time derivative of n and kept leading order in μ terms. The interaction terms in Eq. (5.52) reduce to

$$E + \frac{V \times (\nabla \times A)}{c} - \frac{(1-2\mu)}{enc} J \times (\nabla \times A)\,. \tag{5.54}$$

In Eqs. (5.53) and (5.54) we see the presence of some terms involving μ, in front of $\ddot{D}$ and $J \times B$, respectively. However, in the latter case it occurs in the factor $(1-2\mu)$ and since our ordering is $\mu << 1$, we can drop the μ-dependence in Eq. (5.54), to lowest order. However, in Eq. (5.53), we cannot throw out all the terms that depend on μ since the factor $\mu m_i/(ne^2)$ cannot be cast into a dimensionless form, and hence one cannot invoke the ordering $\mu << 1$ here. Post-variation, the discrepancy in the order of the derived terms, i.e. the existence of these anomalous terms, has also been observed elsewhere [75].

The Eulerian version of the equation of motion of the current Eq. (5.52) (after keeping zeroth order in μ), also known as *generalized Ohm's law*, is then

$$\begin{aligned} E + \frac{V \times B}{c} = \frac{m_e}{e^2 n}\left(\frac{\partial J}{\partial t} + (J\cdot\nabla)V - (J\cdot\nabla)\left(\frac{J}{n}\right) + (\nabla\cdot V)J\right) \\ + \frac{(J\times B)}{enc} - \frac{\nabla p_e}{en} + \frac{m_e}{e^2}(V\cdot\nabla)\left(\frac{J}{n}\right) + \frac{m_e}{e^2 n^2} J(V\cdot\nabla)n\,. \end{aligned} \tag{5.55}$$

The last two terms on the right hand side of Eq. (5.55) can be combined to give $(m_e/(e^2 n))\,(V\cdot\nabla)J$ and since $\nabla\cdot J = 0$, we can add a $V(\nabla\cdot J)$ term without changing the result, and combine most terms in the divergence of the

tensor $VJ + JV$ to obtain the following equation:

$$E+\frac{V\times B}{c}=\frac{m_e}{e^2n}\left(\frac{\partial J}{\partial t}+\nabla\cdot(VJ+JV)\right)-\frac{m_e}{e^2n}(J\cdot\nabla)\left(\frac{J}{n}\right)+\frac{(J\times B)}{enc}-\frac{\nabla p_e}{en}\,. \tag{5.56}$$

Equations (5.51) and (5.56) constitute the extended MHD model.

5.4.2 Hall MHD

Hall MHD is a limiting case, for which previous work of an action functional nature exists [48, 141]. Here we obtain the actional functional by expanding and retaining only terms up to zeroth order in μ, i.e., if we neglect the electron inertia ($m_e \to 0$), the action of Eq. (5.29) reduces to

$$\begin{aligned}
S=&-\frac{1}{8\pi}\int dt\int d^3x\ |\nabla\times A(x,t)|^2\\
&+\int dt\int d^3a\ n_0\left[\frac{1}{2}m|\dot{Q}|^2(a,t)-\mathfrak{U}_i\left(\frac{n_0}{\mathcal{J}_i(Q)},s_{i0}\right)-\mathfrak{U}_e\left(\frac{n_0}{\mathcal{J}_e(Q,D)},s_{e0}\right)\right]\\
&+\int dt\int d^3x\int d^3a\ n_0\left\{\delta\left(x-Q(a,t)-\frac{1}{en_0}D(a,t)\right)\right.\\
&\left.\times\left[-\frac{e}{c}\left(\dot{Q}(a,t)-\frac{1}{en_0}\dot{D}(a,t)\right)\cdot A(x,t)+e\phi(x,t)\right]\right\}\\
&+\int dt\int d^3x\int d^3a\ n_0\left\{\delta\left(x-Q(a,t)\right)\times\left[\frac{e}{c}\dot{Q}(a,t)\cdot A(x,t)-e\phi(x,t)\right]\right\},
\end{aligned} \tag{5.57}$$

and Eqs. (5.41) become

$$\begin{aligned}
q_i(Q,D)&=Q(a,t)\\
q_e(Q,D)&=Q(a,t)-D(a,t)/(en_0)
\end{aligned} \tag{5.58}$$

Observe we have also replaced m_i by m in the kinetic energy term, which is correct to leading order in μ.

The inverse maps required for Eulerianizing the equations of motion reduce to

$$\dot{Q}(a,t) = V(x,t)\Big|_{x=q_i=Q}$$
$$\dot{D}(a,t) = en_0 V(x,t)\Big|_{x=q_i=Q} - en_0\left(V(x',t) - \frac{J(x',t)}{en(x',t)}\right)\Big|_{x'=q_e(Q,D)} . \quad (5.59)$$

Following the procedure outlined in the previous section for extended MHD, we arrive at what is commonly referred to as Hall MHD,

$$nm\left(\frac{\partial V}{\partial t} + (V\cdot\nabla)V\right) = -\nabla p + \frac{J\times B}{c} \quad (5.60)$$
$$E + \frac{V\times B}{c} = \frac{J\times B}{nec} - \frac{1}{ne}\nabla p_e\,, \quad (5.61)$$

which are the usual forms of the momentum equation and Ohm's law for Hall MHD.

5.4.3 Electron MHD

Electron MHD [68, 10, 109, 55] is another limiting case where we neglect the ion motion completely. This theory is used to describe the short time scale motion of the electrons in a neutralizing ion background. Since the ions are immobile, we require $\dot{q}_i = 0$ and $q_i = q_i(a)$. Also, we require that there be no electric field and, consequently, we neglect ϕ from the action. In this case, using the Q, D formulation of the previous sections is redundant since there is only a single fluid. From $\dot{q}_i = 0$ we find $\dot{D} = -\left(en_0 m/m_e\right)\dot{Q}$. (The same relation holds between Q and D up to an additive constant which represents the constant position of the ion). In addition, the Lagrange-Euler map takes

on the simple form

$$v_e(x,t) = \left(1+\frac{1}{\mu}\right)\dot{Q}(a,t)\Big|_{a=q_e^{-1}(x,t)}\,, \tag{5.62}$$

where

$$q_e(a,t) = (1+\frac{1}{\mu})Q(a,t)\,. \tag{5.63}$$

The remaining terms in the action are

$$\begin{aligned} S = &-\frac{1}{8\pi}\int dt \int d^3x\, |\nabla \times A(x,t)|^2 \\ &+ \int dt \int d^3a\; n_0 \left[\frac{1}{2}m_e|\dot{q}_e|^2(a,t) - \mathfrak{U}_e\left(\frac{n_0}{\mathcal{J}_e(Q)}\right)\right] \\ &- \int dt \int d^3x \int d^3a\; \delta\left(x-q_e\right)\frac{en_0}{c}\dot{q}_e\cdot A(x,t)\,, \end{aligned} \tag{5.64}$$

which is essentially the same action as that of Ref. [55]. It is also straightforward to express this action in terms of Q using Eqs. (5.62) and (5.63).

Upon varying the action (either in terms of q_e or Q) and Eulerianizing the following equation of motion and constraint are obtained:

$$\begin{aligned} m_e\left(\frac{\partial v_e}{\partial t} + v_e\cdot\nabla v_e\right) + \frac{e}{c}\frac{\partial A}{\partial t} =& \frac{e}{c}(v_e\times B) - \frac{\nabla p_e}{n} \\ \nabla\times B =& -\frac{4\pi}{c}env_e\,, \end{aligned}$$

which are the usual equations of electron MHD.

5.5 Noether's theorem

In this section we will investigate the invariants of the action functionals for the quasineutral Lüst equations of Eq. (5.29) and the extended MHD

system of Eq. (5.42) using Noether's theorem. Note that both actions can be expressed either in terms of (Q, D) or in terms of (q_i, q_e), which are related through a simple linear transformation, e.g., Eq. (5.27). Furthermore, both sets of variables obey the Eulerian Closure Principle. Hence, it is equivalent to work with an action expressed in terms of either set of variables. For convenience, we shall work with the latter set, as the Euler-Lagrange maps are easier to apply. Noether's theorem states that if an action is invariant under the transformations

$$q'_s = q_s + K_s\left(q_s, t\right); \quad t' = t + \tau\left(t\right), \tag{5.65}$$

i.e.,

$$\begin{aligned} S &= \int_{t_1}^{t_2} dt \int d^3z\ \mathcal{L}\left(q_s, \dot{q}_s, z, t\right) \\ &= \int_{t'_1}^{t'_2} dt' \int d^3z'\ \mathcal{L}\left(q'_s, \dot{q}'_s, z', t'\right)\ , \end{aligned}$$

then there exist constants of motion given by

$$C = \int d^3z \left[\tau\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_s} \cdot \dot{q}_s - \mathcal{L}\right) - K_s \cdot \frac{\partial \mathcal{L}}{\partial \dot{q}_s}\right]\ , \tag{5.66}$$

where the index s represents the number of independent variables q in the system. Our actions are mixed Lagrangian and Eulerian, so the variable z can denote a or x.

1. Time translation

It is straight-forward to see that the action is invariant under time translation with

$$K_s = 0; \quad \tau = 1\,.$$

The corresponding constant of motion, the energy, is found to be

$$\mathcal{E} = \int d^3x \left[\frac{|\nabla \times A|^2}{8\pi} + \sum_s \int d^3a \left(\frac{1}{2} n_0 m_s |\dot{q}_s|^2 + n_0\, \mathfrak{U}_s \left(\frac{n_0}{\mathcal{J}_s}, s_{s0} \right) \right) \right] .$$

Using suitable Lagrange-Euler maps to express our answer in terms of the Eulerian variables $\{n, V, J\}$, we obtain

$$\mathcal{E} = \int d^3x \left[\frac{|B|^2}{8\pi} + n\mathfrak{U}_i + n\mathfrak{U}_e + mn\frac{|V|^2}{2} + \frac{m_e m_i}{mne^2}\frac{|J|^2}{2} \right] , \tag{5.67}$$

for the quasineutral Lüst model and

$$\mathcal{E} = \int d^3x \left[\frac{|B|^2}{8\pi} + n\mathfrak{U}_i + n\mathfrak{U}_e + mn\frac{|V|^2}{2} + \frac{m_e}{ne^2}\frac{|J|^2}{2} \right] , \tag{5.68}$$

for the extended MHD model. Note that the two energies are different since the extended MHD model includes the mass ratio ordering.

2. Space translation

Space translations correspond to

$$K_s = k; \quad \tau = 0\,,$$

where k is an arbitrary constant vector. Under space translations, the constant of motion is the momentum, which is found to be

$$P = k\cdot \int d^3a \left(n_0 m_i \dot{q}_i + n_0 m_e \dot{q}_e \right) + k\cdot \int d^3x\, \frac{e}{c} A \left\{ \int n_0 \left[\delta\left(x - q_i\right) - \delta\left(x - q_e\right) \right] d^3a \right\} .$$

Using the Lagrange-Euler maps one can show that

$$P = k \cdot \int d^3x \; nmV$$

is the conserved quantity. Note that k is entirely arbitrary, and hence we see that the total momentum

$$P = \int d^3x \; \rho V \tag{5.69}$$

is conserved. This is also evident from the corresponding dynamical equation for V.

3. Rotations

The actions are also invariant under rotations which correspond to

$$K_s = k \times q_s; \quad \tau = 0 \, .$$

Following the same procedure as before, we have

$$\mathfrak{L} = k \cdot \int d^3x \; nm \, r \times V \, ,$$

and since we know that k is arbitrary, we conclude that the angular momentum given by

$$\mathfrak{L} = \int d^3x \, \rho \, r \times V \tag{5.70}$$

is a constant of motion.

4. Galilean boosts

When discussing boosts, we have to consider that the action may remain invariant even when the following holds

$$\begin{aligned} S &= \int_{t_1}^{t_2} dt \int d^3z \; \mathcal{L}\left(q_s, \dot{q}_s, z, t\right) \\ &= \int_{t_1'}^{t_2'} dt' \int d^3z' \left(\mathcal{L}\left(q_s', \dot{q}_s', z', t'\right) + \partial_\mu \lambda^\mu\right) , \end{aligned}$$

because the second term vanishes identically. In all the previous derivations of the constants of motion, the infinitesimal transformations did not involve time explicitly. A boost, though, corresponds to

$$K_s = ut; \quad \tau = 0 \, ,$$

where u is an arbitrary constant velocity. For a Galilean boost in a one-fluid model, the corresponding invariant quantity is given by

$$\mathcal{B} = \int d^3a \; mn \left(q - \dot{q}t\right) ,$$

and since we have two different species, this generalizes to

$$\mathcal{B} = \sum_s \int d^3a \; m_s n_s \left(q_s - \dot{q}_s t\right) .$$

Using the corresponding Lagrange-Euler maps, the Eulerianized expression is given by

$$\mathcal{B} = \int d^3x \; \rho \left(x - Vt\right) . \tag{5.71}$$

Chapter 6

Dirac Constraints

In the present chapter, we will present two worked examples of the method of Dirac for imposing constraints on Hamiltonian systems. As we mentioned in Sec. 1.1 oftentimes, we seek reduced descriptions of systems that retain the fundamental physics and we desire these reduced models to retain the Hamiltonian character of the ideal parent model. Now, we will undertake the problem of imposing constraints to a Hamiltonian system in a way that preserves the Hamiltonian character.

The method of Dirac is general [22] and has been applied to finite and infinite-dimentional Hamiltonian systems [99, 100], has been used for the derivation of incompressible fluid dynamics and magnetohydrodynamics [16, 98], reduced fluid models of plasma physics [17], and for the description of dynamics in fluids with free boundaries [91]. The method consists of a procedure that leads to a new bracket for the constrained system, known as the Dirac bracket, which is bilinear, antisymmetric, satisfies the Jacobi identity and furthermore, when one takes the bracket of anyone of the constrains with any other quantity, the bracket vanishes. For an introduction to the method, the interested reader might want to refer to the textbooks of [35], [120] or [119]

and [79]. Here, we will not formally present the method but rather illustrate it using two worked examples relevant to reduced modelling. We will avoid using the notions of strong and weak equality even though they are key and should be understood before we embark on the calculation. To this effect, we define them and we discuss briefly their ramifications. We say that two phase space functions f and g are strongly equal when their equality holds throughout phase space and that they are weakly equal when they are equal only when the constraints are enforced. To make this idea concrete, we add a brief note: Imagine that we have a system of equations for the evolution of density and pressure. In this model, pressure and density are dynamical variables and we have separate, yet coupled, equations that give their time evolution. Now, lets suppose that we want to impose the constraint that the pressure is a specific function of the density, $p = f(n)$ and plug this equality in the equations of motion. Of course, this can only serve as an initial condition since the dynamics will evolve both p and n independently and after a while, in general $p \neq f(n)$. This would be an example of a weak equality. If we want the condition $p = f(n)$ to hold for all times, i.e, to be a strong equality, then we need to impose it as a constraint. In that event, the Dirac method will give us a new bracket and new equations of motion where additional terms will appear. They should be understood as generalized forces, that constrain the original system to evolve in such a way as to preserve the constraints.

6.1 Dirac Constraints on the Hazeltine Model

6.1.1 Definitions

For the first worked example of the Dirac method we choose a model initially presented in Ref. [38] and shown to be Hamiltonian in Ref. [39]. The model was used for the study of electromagnetic solitary drift waves. It is a three field model involving the electrostatic potential ϕ, the parallel component of the magnetic flux ψ and a variable χ which is defined as the departure of the plasma density n from a constant value n_c, or in other words, the density perturbation. Also, we introduce the variables U and J which stand for the vorticity and the parallel current density respectively, through the relations:

$$U = \nabla_\perp^2 \phi \,, \tag{6.1}$$

$$J = \nabla_\perp^2 \psi \,. \tag{6.2}$$

In terms of these fields, the system of equations, when appropriate normalizations have been made, is given by:

$$\frac{\partial U}{\partial t} + [\phi, U] + \nabla_\parallel J = 0 \,, \tag{6.3}$$

$$\frac{\partial \psi}{\partial t} + \nabla_\parallel \phi = \eta J + \alpha \nabla_\parallel \chi \,, \tag{6.4}$$

$$\frac{\partial \chi}{\partial t} + [\phi, \chi] + \nabla_\parallel J = 0 \,, \tag{6.5}$$

where $\nabla_\parallel f = \partial f / \partial z - [\Psi, f]$, with $[\cdot, \cdot]$ denoting the canonical Poisson bracket, so that $[f, g] = \hat{\mathbf{z}} \cdot (\nabla f \times \nabla g)$. Eq. (6.3) is the vorticity evolution equation and is identical to the RMHD one. Eq. (6.4) is a generalized Ohm's law where the parallel component of the electric field is equated to the parallel current

plus the pressure gradient, where here, because the model is isothermal, is only given as the gradient of the pressure perturbation. η here is a normalised collisional resistivity which we will later drop as we are only interested in the non-dissipative dynamics. α is a parameter defined as the squared ratio of the ion gyroradius to transverse scale lengths, $\alpha = \dfrac{\rho_s^2}{L}$. By transverse scale lengths we mean the length scale of variation of quantities perpendicular to the magnetic field, i.e., $\nabla^{-1} \sim L$. Finally, Eq. (6.5) describes the conservation of the density. The second term is the usual $E \times B$ advection term while the third one implies that the divergence of the electron parallel flow is the total parallel current divergence since ion flow is very small.

The Hamiltonian of this system, as given in Ref. [39] is:

$$H = \frac{1}{2} \int d^3x \left((\nabla_\perp \psi)^2 + (\nabla_\perp \phi)^2 + \alpha \chi^2 \right) , \tag{6.6}$$

where we can clearly discern the terms for magnetic field energy, electrostatic field energy and kinetic energy.

The Lie-Poisson bracket for the above system is:

$$\begin{aligned} \{F, G\} = \int d^2x \{ & U[F_U, G_U] + \chi[F_\chi, G_\chi] \\ & + \psi([F_\psi, G_U + G_\chi] - [G_\psi, F_U + F_\chi]) \\ & + \chi([F_\chi, G_U] - [G_\chi, F_U]) \} . \end{aligned} \tag{6.7}$$

Even though the system is three-dimensional, for the rest of this section, we focus only on it's two-dimensional limit by discarding the terms $\frac{\partial \phi}{\partial z}$, $\frac{\partial \chi}{\partial z}$, and $\frac{\partial J}{\partial z}$. This is the reason that we present only the two-dimensional version of the Lie-Poisson bracket.

6.1.2 Constraints and change of variables

As it can be immediately observed [38], the above model reduces to the low-β RMHD one when α is neglected. This is appropriate when the transverse perturbations have a scale length much larger than the gyroradius. Also, the vorticity equation Eq. (6.3), reduces to the famous Hasegawa-Mima [36] equation when the electrons are assumed to be adiabatic, i.e, $\alpha\chi = \phi$. Because this is an equation involving dynamical fields, as we know from the introduction to this chapter, it can only be prescribed as an initial condition. Here, we are interested in imposing electron adiabaticity as a Dirac constraint and recovering the constrained dynamical equations through the Dirac procedure. Because the method of Dirac usually works with an even number of constraints[1], we need one more constraint. For our other constraint, we will choose to impose that the magnetic flux is only a function of x. This is indeed a very strong constraint yet not completely unphysical. We can imagine a scenario where the magnetic field is imposed externally by a coil configuration which has an active, real-time feedback loop that regulates the current in such a way as to preserve the flux constraint. Certainly, this kind of control will induce generalized forces that will force the system to develop on the curve of the constraint. Therefore, the two constraints we want to impose are:

$$\Phi_1(\mathbf{x}) = U - \alpha\Delta\chi = 0\,, \tag{6.8}$$

$$\Phi_2(\mathbf{x}) = \psi - f(x) = 0\,. \tag{6.9}$$

[1]The reader who is interested in finding examples of the Dirac method working with an odd number of constraints is referred to Ref. [14]

To simplify the calculations to follow, we make the following change of variables:

$$\overline{U} = \chi\,, \tag{6.10}$$

$$\overline{\psi} = \psi\,, \tag{6.11}$$

$$\overline{\chi} = U - \chi\,. \tag{6.12}$$

With this change of variables, the variational derivatives of the Hamiltonian (6.6) with respect to the new fields become:

$$H_{\overline{U}} = \alpha\overline{U} - \overline{\phi}\,, \tag{6.13}$$

$$H_{\overline{\psi}} = -\overline{J}\,, \tag{6.14}$$

$$H_{\overline{\chi}} = -\overline{\phi}\,. \tag{6.15}$$

where $\nabla^2\overline{\phi} = \overline{U} + \overline{\chi}$ and $\nabla^2\overline{\psi} = \overline{J}$ and the transformed equations of motion Eq. (6.3) – Eq. (6.5) are:

$$\dot{\overline{U}} = [\overline{U}, \overline{\phi}] + [\overline{\psi}, \overline{J}]\,, \tag{6.16}$$

$$\dot{\overline{\psi}} = [\overline{\psi}, \overline{\phi} - \alpha\overline{U}]\,, \tag{6.17}$$

$$\dot{\overline{\chi}} = [\overline{\chi}, \overline{\phi}]\,. \tag{6.18}$$

The Lie-Poisson bracket (6.7) becomes:

$$\{F, G\} = \int d^2x \left\{\overline{U}[F_{\overline{U}}, G_{\overline{U}}] + \overline{\chi}[F_{\overline{\chi}}, G_{\overline{\chi}}] + \overline{\psi}\left([F_{\overline{\psi}}, G_{\overline{U}}] + [F_{\overline{U}}, G_{\overline{\psi}}]\right)\right\}\,, \tag{6.19}$$

and the new constraints are given by:

$$\Phi_1(\mathbf{x}) = \overline{U} + \overline{\chi} - \alpha\Delta\overline{U} = 0\,, \tag{6.20}$$

$$\Phi_2(\mathbf{x}) = \overline{\psi} - f(x) = 0\,. \tag{6.21}$$

For the calculations that will follow, it will prove useful to define the operators: $L_\Delta = -1 + \alpha\Delta$, $L_\xi = [\xi, \cdot]$ where ξ can be any of the $\overline{U}, \overline{\psi}, \overline{\chi}$. Then, the operator L_Δ is self-adjoint while all the others are anti-self-adjoint. We also remind that when an operator is primed, it means that it acts on the x' variable.

Finally, we note that in terms of L_Δ, the first constraint can be expressed as: $\Phi_1(\mathbf{x}) = \overline{\chi} - L_\Delta\overline{U} = 0$.

6.1.3 The Dirac Method

The Dirac bracket for the constrained system is given by:

$$[F, G]_{DB} = \{F, G\}_{PB} - \iint d^2x d^2x' \{F, \Phi_a(\mathbf{x})\} C^{-1}_{ab}(\mathbf{x}, \mathbf{x}') \{\Phi_b(\mathbf{x}'), G\}\,. \tag{6.22}$$

where C^{-1}_{ab} is the inverse of the constrain matrix C, defined as $C_{ab}(\mathbf{x}, \mathbf{x}') = \{\Phi_a(\mathbf{x}), \Phi_b(\mathbf{x}')\}$, with $\{\cdot, \cdot\}$ being the Lie-Poisson bracket of the unconstrained system. In the following, we calculate explicitly all terms needed in order to apply Eq. (6.22).

6.1.3.1 Calculation of Matrix Elements

From now on, we drop the overbar notation. All field variables should be understood being the barred ones and in the end, we will switch again to

the unbarred ones. First, we calculate the constraint matrix elements:

$$
\begin{aligned}
C_{11}(\mathbf{x},\mathbf{x}') =& \{\Phi_1(\mathbf{x}),\Phi_1(\mathbf{x}')\} \\
=& \int d^2x''\{U(x'')[\delta(x-x'')-\alpha\Delta\delta(x-x''),\delta(x'-x'')-\alpha\Delta'\delta(x'-x'')] \\
&+\chi(x'')[\delta(x-x''),\delta(x'-x'')]\} \\
=& \int d^2x''\{-\delta(x-x'')[U(x''),\delta(x'-x'')]+\delta(x-x'')\alpha\Delta'[U(x''),\delta(x'-x'')] \\
&+\delta(x-x'')\alpha\Delta[U(x''),\delta(x'-x'')]-\delta(x-x'')\alpha^2\Delta\Delta'[U(x''),\delta(x'-x'')] \\
&-\delta(x-x'')[\chi(x''),\delta(x'-x'')]\} \\
=& -[U,\delta(x'-x)]+\alpha\Delta'[U,\delta(x'-x)]+\alpha\Delta[U,\delta(x'-x)] \\
&-\alpha^2\Delta\Delta'[U,\delta(x'-x)]-[\chi,\delta(x'-x)] \\
=& \left(-L_U+\alpha\Delta'L_U+\alpha\Delta L_U-\alpha^2\Delta\Delta'L_U-L_\chi\right)\delta(x'-x)\,, \qquad (6.23)
\end{aligned}
$$

where in the first step we use integration by parts, in the second step we evaluate the delta functions and in the last one, we re-express the result in compact form, using the operators defined at the end of the previous section. Furthermore, we observe that $L_\Delta L_{\Delta'} = (-1+\alpha\Delta)(-1+\alpha\Delta') = (1-\alpha\Delta-\alpha\Delta'+\alpha^2\Delta\Delta')$ which allows us to write the result in the form:

$$
C_{11}(\mathbf{x},\mathbf{x}') = -\left(L_\Delta L_U L_{\Delta'}+L_\chi\right)\delta(x'-x)\,, \qquad (6.24)
$$

since L_U and $L_{\Delta'}$ commute because they are acting on different variables.

We proceed by calculating C_{12}:

$$\begin{aligned} C_{12}(\mathbf{x},\mathbf{x}') =& \{\Phi_1(\mathbf{x}),\Phi_2(\mathbf{x}')\} \\ =& \int d^2x'' \left\{\psi(x'')[\delta(x-x'')-\alpha\Delta\delta(x-x''),\delta(x'-x'')]\right\} \\ =& \int d^2x'' \{-\delta(x-x'')[\psi(x''),\delta(x'-x'')]+\delta(x-x'')\alpha\Delta[\psi(x''),\delta(x'-x'')] \\ =& (-1+\alpha\Delta)[\psi,\delta(x'-x)]\,. \end{aligned} \tag{6.25}$$

Expressed in terms of the L-operators:

$$C_{12}(\mathbf{x},\mathbf{x}') = L_\Delta L_\psi \delta(x'-x) = L_\Delta L_\psi \delta(x-x')\,. \tag{6.26}$$

The calculation of $C_{21}(\mathbf{x},\mathbf{x}')$ is exactly the same and we will not repeat it but simply state the result:

$$C_{21}(\mathbf{x},\mathbf{x}') = -L_{\Delta'} L_{\psi'} \delta(x-x')\,, \tag{6.27}$$

which can be easily verified using the symmetry relation:

$$C_{ab}(\mathbf{x},\mathbf{x}') = -C_{ba}(\mathbf{x}',\mathbf{x})\,. \tag{6.28}$$

which follows from the antisymmetry of the Poisson bracket.

Finally, it is straightforward to show that:

$$C_{22}(\mathbf{x},\mathbf{x}') = 0\,. \tag{6.29}$$

6.1.3.2 Calculation of Inverse Matrix Elements

To calculate the inverse matrix elements we use the identity:

$$\int dx'' C_{ik}(\mathbf{x},\mathbf{x}'') C^{-1}_{kj}(\mathbf{x}'',\mathbf{x}') = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}')\,. \tag{6.30}$$

We start with C_{12}^{-1}:

$$\begin{aligned}\int dx'' C_{21}(x,x'')C_{12}^{-1}(x'',x') + \int dx'' C_{22}(x,x'')C_{22}^{-1}(x'',x') &= \delta(x-x') \\ -\int dx'' L_{\Delta''}L_{\psi''}\delta(x-x'')C_{12}^{-1}(x'',x') &= \delta(x-x') \\ L_\psi L_\Delta C_{12}^{-1}(x,x') &= \delta(x-x')\,,\end{aligned} \tag{6.31}$$

where in the second step we integrated by parts twice using the fact that L_Δ is self-adjoint and L_ψ is anti self-adjoint and in the last step, we evaluated the delta function. C_{12}^{-1} can be now calculated by inverting the result of the last step (assuming L_Δ^{-1},L_ψ^{-1} exist):

$$C_{12}^{-1}(\mathbf{x},\mathbf{x}') = L_\Delta^{-1}L_\psi^{-1}\delta(x-x')\,. \tag{6.32}$$

Eq. (6.30) implies the symmetry relation:

$$C_{ab}^{-1}(\mathbf{x},\mathbf{x}') = -C_{ba}^{-1}(\mathbf{x}',\mathbf{x})\,, \tag{6.33}$$

for the inverse matrix elements. With the help of the above:

$$C_{21}^{-1}(\mathbf{x},\mathbf{x}') = -L_{\Delta'}^{-1}L_{\psi'}^{-1}\delta(x'-x) = -L_{\Delta'}^{-1}L_{\psi'}^{-1}\delta(x-x')\,. \tag{6.34}$$

For the calculation of C_{22}^{-1} we will use the following trick: We assume $C_{22}^{-1}(x,x') = \mathcal{X}\delta(x-x')$ where $\mathcal{X}$ is some product of operators that we want

to specify. Next, we substitute in Eq. (6.30) as follows:

$$\begin{aligned}
\int dx'' C_{12}(x,x'')C_{22}^{-1}(x'',x') &= -\int dx'' C_{11}(x,x'')C_{12}^{-1}(x'',x') \\
\int dx'' L_\Delta L_\psi \delta(x-x'')\mathcal{X}\delta(x''-x') &= -\int dx'' C_{11}(x,x'')L_{\Delta''}^{-1}L_{\psi''}^{-1}\delta(x''-x') \\
-\int dx'' L_\Delta L_{\psi''} \delta(x-x'')\mathcal{X}\delta(x''-x') &= \int dx'' L_{\psi''}^{-1}L_{\Delta''}^{-1}C_{11}(x,x'')\delta(x''-x') \\
-\int dx'' L_{\psi''} L_\Delta \delta(x-x'')\mathcal{X}\delta(x''-x') &= L_{\psi'}^{-1}L_{\Delta'}^{-1}C_{11}(x,x') \\
\int dx'' \mathcal{X}^\dagger L_\psi L_\Delta \delta(x-x'')\delta(x''-x') &= L_{\psi'}^{-1}L_{\Delta'}^{-1}C_{11}^{*}\delta(x-x') \\
\mathcal{X}^\dagger L_\psi L_\Delta \delta(x-x') &= L_{\psi'}^{-1}L_{\Delta'}^{-1}C_{11}^{*}\delta(x-x')\,, \qquad (6.35)
\end{aligned}$$

where by $C_{11}^{*}(x,x')$ we denote the operator part of $C_{11}(x,x')$. A few words are in order about this calculation. Going from the second to the third line we have used the property $L_\psi\delta(x-x') = -L_{\psi'}\delta(x-x')$, which is elementary to prove, given that L_ψ involves only first derivatives. In the next line, we interchange the L_ψ and L_Δ operators since now, they act on different arguments (therefore they commute) and immediately afterwards, we change back the argument on which L_ψ acts. Also, we make the $\mathcal{X}$ operator act on the first delta function, and in the last line, we evaluate the remaining delta function. One may wonder, why do we have to go through all these steps and not arrive at the result in a more straighforward way. The answer to this question is that, with hindsight, we are looking for a particular form for our inverse element. It is well known that infinite-dimentional operators may not have unique inverses. Any other sequence of steps would yield a different result. Having tried several of these, it appears that all the subsequent steps of the calculation simplify considerably

when we choose the particular result that arises from Eq. (6.35). In addition, one needs to be careful about integrating $\mathcal{X}$ by parts. Since $\mathcal{X}$ is a product of operators with some acting on primed and some acting on unprimed variables, first we need to make all of them act on the integration variable, using the same trick we used before in the case of L_ψ. After we do that, we can integrate by parts. Here, we have assumed that all of these operations (changing the acting variable and integrating by parts), will add up to give us a positive sign. After we calculate the exact form for $\mathcal{X}$ we can count the sign flips needed for the described procedure, to confirm our choice. With all this in mind, we arrive at $\mathcal{X} = L_\psi^{-1} L_\Delta^{-1} C_{11}^{*\dagger} L_{\Delta'}^{-1} L_{\psi'}^{-1}$, which yields:

$$C_{22}^{-1}(\mathbf{x}, \mathbf{x}') = L_\psi^{-1} L_\Delta^{-1} C_{11}^{*\dagger} L_{\Delta'}^{-1} L_{\psi'}^{-1} \delta(x - x') \,. \tag{6.36}$$

To conclude with this calculation, note that the resulting form of $\mathcal{X}$ suggests a number of four sign flips for the whole process of integrating by parts, which confirms our initial guess for the choice of sign. Finally, it is easily verified that:

$$C_{11}^{-1}(\mathbf{x}, \mathbf{x}') = 0 \,. \tag{6.37}$$

6.1.3.3 Calculation of Integral Terms

We proceed by computing the integral terms of Eq. (6.22). To do that, we first need the Poisson brackets of the constraints with any functional F.

The calculation is straightforward and the result is:

$$\{F, \Phi_1\} = -L_\Delta L_U F_U - L_\Delta L_\psi F_\psi + L_\chi F_\chi \,, \tag{6.38}$$

$$\{\Phi_1, G\} = L_{\Delta'} L_{U'} G_U + L_\Delta L_{\psi'} G_\psi - L_{\chi'} G_\chi \,, \tag{6.39}$$

$$\{F, \Phi_2\} = L_\psi F_U \,, \tag{6.40}$$

$$\{\Phi_2, G\} = -L_{\psi'} G_U \,. \tag{6.41}$$

We have only three non-zero inverse matrix elements so we calculate three integral terms:

$$\begin{aligned}
&\iint d^2x d^2x' \{F, \Phi_1\} C_{12}^{-1}(\mathbf{x}, \mathbf{x}') \{\Phi_2, G\} \\
&= \iint d^2x d^2x' \left(-L_\Delta L_U F_U - L_\Delta L_\psi F_\psi + L_\chi F_\chi\right) L_\Delta^{-1} L_\psi^{-1} \delta(x - x') \left(-L_{\psi'} G_U\right) \\
&= \int d^2x \left(-L_\psi^{-1} L_U F_U - F_\psi + L_\psi^{-1} L_\Delta^{-1} L_\chi F_\chi\right) \left(L_\psi G_U\right) \\
&= \int d^2x \left(-F_U L_U G_U - F_\psi L_\psi G_U + F_\chi L_\chi L_\Delta^{-1} G_U\right) \,.
\end{aligned} \tag{6.42}$$

$$\begin{aligned}
&\iint d^2x d^2x' \{F, \Phi_2\} C_{21}^{-1}(\mathbf{x}, \mathbf{x}') \{\Phi_1, G\} \\
&= -\iint d^2x d^2x' \left(L_\psi F_U\right) L_{\Delta'}^{-1} L_{\psi'}^{-1} \delta(x - x') \left(L_{\Delta'} L_{U'} G_U + L_{\Delta'} L_{\psi'} G_\psi - L_{\chi'} G_\chi\right) \\
&= \int d^2x \left(L_\psi F_U\right) \left(L_\psi^{-1} L_U G_U + G_\psi - L_\psi^{-1} L_\Delta^{-1} L_\chi G_\chi\right) \\
&= \int d^2x \left(-F_U L_U G_U - F_U L_\psi G_\psi + F_U L_\Delta^{-1} L_\chi G_\chi\right) \,.
\end{aligned} \tag{6.43}$$

$$
\begin{aligned}
&\iint d^2x d^2x' \{F, \Phi_2\} C_{22}^{-1}(\mathbf{x}, \mathbf{x}') \{\Phi_2, G\} \\
&= - \iint d^2x d^2x' (L_\psi F_U) L_\psi^{-1} L_\Delta^{-1} (L_\Delta L_U L_{\Delta'} + L_\chi) L_{\Delta'}^{-1} L_{\psi'}^{-1} \delta(x - x') (L_{\psi'} G_U) \\
&= \int d^2x (L_\psi^{-1} L_U F_U + L_\psi^{-1} L_\Delta^{-1} L_\chi L_\Delta^{-1} F_U)(L_\psi G_U) \\
&= \int d^2x (F_U L_U G_U + F_U L_\Delta^{-1} L_\chi L_\Delta^{-1} G_U) \,. \qquad (6.44)
\end{aligned}
$$

In all the above calculations, in the third line we always take the adjoint of the operator in front of the delta function and simultaneously, evaluate the delta function. That's why we end up with operators acting only on unprimed variables. In the last step, we present the result by making all operators act on the G functional. The reason for that is that in the next step when we will evaluate the constrained systems equations of motion, we will have the fileds taking up the first slot in the Dirac bracket and the Hamiltonian taking up the second. As a result, we want to have the first slot, free of operators so that when we plug in our fields, we will only get zeros and delta functions that will help us evaluate the integrals.

6.1.3.4 Calculating New Equations of Motion

Now, we have everything we need to calculate the equations of motion of the constrained system. The full Dirac bracket is rather large and not very useful by itself to write down. As we mentioned, we arrive at it by following the recipe of Eq. (6.22). To calculate the seperate equations of motion though, we don't need all the terms. We only need those terms whose first slot, i.e,

the F functional, has a variation with respect to the field whose equation of motion we calculate. Otherwise, the result of the variation will be zero and the term would be irrelevant. Also, just so that we don't burden the reader with unnecessary algebra, all the terms that come from the original Poisson bracket are taken for granted and we only subtract from them the relevant terms of the integral terms.

We start with the ψ-equation:

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= [\psi, H]_{DB} \\ &= [\psi, \phi - \alpha U] + \int dx\, (\delta(x)[\psi, \alpha U - \phi]) \\ &= 0\,. \end{aligned} \tag{6.45}$$

The first term comes from the Poisson bracket while the second is the only term from the integral terms that has an F_ψ. Evidently, this is automatically evaluated to give a delta function, the bracket operator (L_ψ in this case) is written out as a bracket and in the second slot, we plug in the variational derivative of the Hamiltonian Eq. (6.14).

We see that the result comes out to be as expected. The second constraint prescribes ψ to be a certain function of x, thus it can't change by the dynamics.

Next, we move to χ-equation:

$$\begin{aligned}\frac{\partial \chi}{\partial t} &= [\chi, H]_{DB} \\ &= [\chi, \phi] - \int dx \left(\delta(x)[\chi, \alpha L_\Delta^{-1} U - L_\Delta^{-1}\phi]\right) \\ &= [\chi, \phi] - \alpha[\chi, L_\Delta^{-1} U] + [\chi, L_\Delta^{-1}\phi]\,. \end{aligned} \tag{6.46}$$

We observe the presence of new terms in the right hand side of the evolution equation of χ. These new terms, amount to the generalized forces of which we talked earlier, that force the system to preserve the constraints.

Finally, we calculate the U-equation:

$$\begin{aligned}\frac{\partial U}{\partial t} &= [U, H]_{DB} \\ &= [U, \phi] + [\psi, J] + \int dx(\delta(x)[U, \alpha U - \phi]) \\ &+ \int dx(\delta(x)[U, \alpha U - \phi]) + \int dx(\delta(x)[\psi, -J]) - \int dx(\delta(x) L_\Delta^{-1}[\chi, -\phi]) \\ &- \int dx(\delta(x)[U, \alpha U - \phi]) - \int dx(\delta(x) L_\Delta^{-1}[\chi, L_\Delta^{-1}\alpha U - L_\Delta^{-1}\phi]) \\ &= [U, \phi] + [\psi, J] - [U, \phi] - [U, \phi] - [\psi, J] \\ &+ L_\Delta^{-1}[\chi, \phi] + [U, \phi] - \alpha L_\Delta^{-1}[\chi, L_\Delta^{-1} U] + L_\Delta^{-1}[\chi, L_\Delta^{-1}\phi] \\ &= L_\Delta^{-1}[\chi, \phi] - \alpha L_\Delta^{-1}[\chi, L_\Delta^{-1} U] + L_\Delta^{-1}[\chi, L_\Delta^{-1}\phi]\,. \end{aligned} \tag{6.47}$$

Here, more generalized forces come into play. Their physical interpretation is not obvious and indeed, one might question the usefulness of obtaining one. From a mathematical standpoint though, we are content that this result agrees with our first constraint. Since $\Phi_1(x) = \chi - L_\Delta U$ we expect that

$\dot{\Phi}_1(x) = \dot{\chi} - L_\Delta \dot{U}$. It is easily seen that if we apply $-L_\Delta$ to our U-equation and add the result to the χ-equation, we get zero as we expected.

To conclude this subsection, we should remember that all the variables in the previous subsections were overbar variables. Now, we turn our results into our initial variables:

$$\dot{U} = \dot{\chi} + [U - \chi, \phi] - \alpha[U - \chi, L_\Delta^{-1}\chi] + [U - \chi, L_\Delta^{-1}\phi]\,, \tag{6.48}$$

$$\dot{\chi} = L_\Delta^{-1}[U - \chi, \phi] - \alpha L_\Delta^{-1}[U - \chi, L_\Delta^{-1}\chi] + L_\Delta^{-1}[U - \chi, L_\Delta^{-1}\phi]\,. \tag{6.49}$$

6.1.4 Dispersion Relations

As we saw in the previous subsection, the initial Hazeltine model, when constrained such that the electrons are adiabatic and magnetic flux is prescribed, doesn't reduce to the Hasegawa-Mima equation but to something much more complicated, Eq. (6.48)-Eq. (6.49) where generalized forces perform the task of imposing the constraints. Therefore, we would like to know what sort of modes does this new, constrained system possess and how do they differ from the modes of the original one. To find out, we perform a linear analysis of the dispersion relations assuming linear equilibria $\phi_{eq} = \lambda x$, $\psi_{eq} = \mu x$ which imply a linear equilibrium in χ as well, $\chi = \dfrac{\lambda}{\alpha} x$. We also assume small perturbations of the fields of the form:

$$\delta\Phi = \sum_{k=0}^{\infty} \Phi_k e^{i(\omega t - \mathbf{x}\cdot\mathbf{k})}\,. \tag{6.50}$$

We linearize the original system around the aforementioned equilibria and find the dispersion relation to be:

$$\omega^3 - 2k_y\lambda\omega^2 + (k_y^2(1-\mu)(1+\mu) - \alpha k_\perp^2 k_y^2\mu^2)\omega + \alpha\mu^2 k_\perp^2 k_y^3 = 0\,. \qquad (6.51)$$

This equation has three roots and none of them is zero. It is interesting to look at the limits of Eq. (6.51):

- $$\alpha = 0$$

  This is the RMHD limit which always gives a zero mode and the mode

  $$\omega = \lambda k_y \pm k_y\sqrt{\lambda^2 + \mu^2 - 1}$$

  which for $\lambda = 0$ gives an Alfvén wave and for $\mu = 0$, gives just a Doppler shifted wave.

- $$\alpha \neq 0$$

  This is the Hasegawa-Mima limit which gives rise to modes that can be identified with drift waves and for $\mu = 0$ gives only an Alfvén wave $\omega = 2k_y\lambda$.

Now, we proceed to give the dispersion relation of the constrained system which is:

$$\omega^2 = \left(\frac{1-\alpha}{\alpha}\frac{k_y\lambda}{(1+\alpha k_\perp^2)} + \frac{k_y\lambda}{(1+\alpha k_\perp^2)}(2+\alpha k_\perp^2) + \frac{1-\alpha}{\alpha}\frac{\partial L_\Delta^{-1}x}{\partial x}k_y\lambda\right)\omega\,. \qquad (6.52)$$

This one contains a zero mode and a modified drift wave

$$\omega = \left(\frac{1-\alpha}{\alpha} \frac{k_y \lambda}{(1+\alpha k_\perp^2)} + \frac{k_y \lambda}{(1+\alpha k_\perp^2)} (2 + \alpha k_\perp^2) \right) ,$$

which for $\lambda = 0$ it collapses only to the zero mode.

Here, we should add a remark about why we put the term

$$\frac{\partial L_\Delta^{-1} x}{\partial x}$$

equal to zero. We have:

$$\frac{\partial L_\Delta^{-1} x}{\partial x} = \frac{\partial}{\partial x} \iint dx' dy' G(x, x', y, y') x' = \iint dx' dy' \frac{\partial G(x, x', y, y')}{\partial x} x' =$$

$$- \iint dx' dy' \frac{\partial G(x, x', y, y')}{\partial x'} x' = \iint dx' dy' G(x, x', y, y') = \frac{\delta(k)}{-1 + a k_y^2} = 0 ,$$

$$\forall k \neq 0 .$$

where in the last step we Fourier transformed and in the step before that, we integrated by parts.

To sum up, we see that the constrained system contains a drift wave mode, albeit modified. As this was an educational project on the Dirac Constraints method, it wasn't in our intentions to investigate the properties of the constrained system in more detail.

6.2 Dirac Constraints on the two-fluid model

The second worked example of the Dirac method is an attempt to impose quasineutrality on the two-fluid model that we saw in Sec. 5.2. This work was done in collaboration with Manasvi Lingam.

6.2.1 The two-fluid bracket

We know from Sec. 5.2 that the N-fluid action is comprised of three different components:

- **I.** The particle component - the kinetic and internal energy densities of the fluid particles.
- **II.** The field component - the electric and magnetic energy densities of the EM fields.
- **III.** The particle-field interaction component - the Lagrangian for a particle in an electromagnetic field is generalized to a continuum model.

Hence, it is reasonable to assume that a N-fluid bracket must also exhibit the same properties. Let us consider each of them in turn.

For Part I, we know that this is essentially the same as that of a neutral fluid, albeit for each species. Hence, the bracket is

$$\{F,G\}_I = -\sum_s \int \Big(\mathbf{M}_s \cdot \left[\left(F_{\mathbf{M}_s} \cdot \nabla\right) G_{\mathbf{M}_s} - \left(G_{\mathbf{M}_s} \cdot \nabla\right) F_{\mathbf{M}_s} \right] + \rho_s \left[\left(F_{\mathbf{M}_s} \cdot \nabla\right) G_{\rho_s} - \left(G_{\mathbf{M}_s} \cdot \nabla\right) F_{\rho_s} \right] + \sigma_s \left[\left(F_{\mathbf{M}_s} \cdot \nabla\right) G_{\sigma_s} - \left(G_{\mathbf{M}_s} \cdot \nabla\right) F_{\sigma_s} \right] \Big) d^3x \,. \tag{6.53}$$

In the above expression, ρ_s denotes the mass density of each species, $\mathbf{M}_s = \rho_s \mathbf{v}_s$ is the momentum density of each species and σ_s is the entropy density of each species.

For Part II, the bracket for the EM fields is well known, and is given by

$$\{F, G\}_{II} = -\int \left[F_{\mathbf{E}} \cdot (\nabla \times G_{\mathbf{B}}) - G_{\mathbf{E}} \cdot (\nabla \times F_{\mathbf{B}})\right] d^3x \,. \tag{6.54}$$

For Part III, a mixture of guesswork and analysis is used, and it yields

$$\{F, G\}_{III} = \sum_s \int \kappa_s \rho_s \left[(F_{\mathbf{M}_s} \cdot G_{\mathbf{E}} - G_{\mathbf{M}_s} \cdot F_{\mathbf{E}}) + \mathbf{B} \cdot (F_{\mathbf{M}_s} \times G_{\mathbf{M}_s})\right] d^3x \,. \tag{6.55}$$

Note that $\kappa_s = q_s/m_s$, where the q's and m's represent the charges and the masses of each species. The total bracket is given by

$$\{F, G\} = \{F, G\}_I + \{F, G\}_{II} + \{F, G\}_{III} \,, \tag{6.56}$$

and first appeared in Ref. [114].

The Hamiltonian should be a sum of the kinetic and internal energy of the fluid species and the energy in the electromagnetic fields:

$$H = \int \left(\sum_s \left[\frac{|\mathbf{M}_s|^2}{2\rho_s} + \rho_s U_s \left(\rho_s, \sigma_s\right) \right] + \frac{|\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2} \right) d^3x \,. \tag{6.57}$$

The dynamical variables with which we will proceed are ρ_s, σ_s, $\mathbf{M}_s$ and the EM fields $\mathbf{E}$ and $\mathbf{B}$.

6.2.2 The Dirac Method on the 2-fluid bracket

6.2.2.1 Constraints

The two constraints we want to implement are:

$$\Phi_1(\mathbf{x}) = \nabla \cdot \mathbf{E} = 0 \,, \tag{6.58}$$

$$\Phi_2(\mathbf{x}) = \nabla \cdot \mathbf{J} = 0 \,. \tag{6.59}$$

Eq. (6.58) expresses quasineutrality. We have already seen that quasineutrality is a property of charged fluids when we are interested in length scales larger than the Debye length. The second constraint states simply that the current needs to be divergence free, i.e, there are no sources or sinks of current within the volume that the fluids occupy. We note that the constraint $\nabla \cdot \mathbf{B} = 0$ is satisfied automatically.

6.2.2.2 Constraint Matrix

First we need to calculate the constraint matrix, C with $C_{ij}(\mathbf{x}, \mathbf{x}') = \{\Phi_i(\mathbf{x}), \Phi_j(\mathbf{x}')\}$. This time, we won't show explicitly all the steps of the calculation.

We have the following results:

$$\begin{aligned} C_{22}(\mathbf{x}, \mathbf{x}') = &- e^2 \left[\nabla'_j \nabla'_i \mathbf{M}_j(x') \nabla'_i + \nabla_j \nabla_i \mathbf{M}_j(x) \nabla_i\right] \delta(x - x') \\ &+ e^3 \left(\frac{1}{m_i^3} - \frac{1}{m_e^3}\right) \nabla' \cdot [\mathbf{B}(x') \times \nabla \delta(x - x')] \,, \end{aligned} \tag{6.60}$$

$$C_{12}(\mathbf{x}, \mathbf{x}') = e^2 \nabla \cdot [N(x) \nabla \delta(x - x')] = e^2 \mathcal{L} \delta(x - x') \,, \tag{6.61}$$

$$C_{21}(\mathbf{x}, \mathbf{x}') = -e^2 \nabla' \cdot [N(x') \nabla' \delta(x - x')] = -e^2 \mathcal{L}' \delta(x - x') \,, \tag{6.62}$$

$$C_{11}(\mathbf{x},\mathbf{x}') = 0\,, \tag{6.63}$$

where we have defined the vector $\mathbf{M}(x) = \left(\frac{\mathbf{M}_i(x)}{m_i^2} + \frac{\mathbf{M}_e(x)}{m_e^2}\right)$, the function $N(x)$ as $N(x) = \left(\frac{n_i(x)}{m_i} + \frac{n_e(x)}{m_e}\right)$ and the operator $\mathcal{L} = \nabla N(x)\cdot\nabla + N(x)\nabla^2$. We observe that Eqs. (6.61)-(6.62) satisfy the symmetry relation of Eq. (6.28).

6.2.2.3 Inverse Matrix Elements

Using the above results, we calculate the inverse matrix elements using Eq. (6.30). We get

$$C_{12}^{-1}(\mathbf{x},\mathbf{x}') = \nabla^{-1}\left(\frac{\nabla^{-1}\delta(x-x')}{e^2N(x)}\right) = \frac{1}{e^2}\mathcal{L}^{-1}\delta(x-x')\,, \tag{6.64}$$

$$C_{21}^{-1}(\mathbf{x},\mathbf{x}') = -\nabla'^{-1}\left(\frac{\nabla'^{-1}\delta(x'-x)}{e^2N(x')}\right) = -\frac{1}{e^2}\mathcal{L}'^{-1}\delta(x-x')\,, \tag{6.65}$$

$$\begin{aligned}\nabla\cdot\left[N(x)\nabla C_{22}^{-1}(\mathbf{x},\mathbf{x}')\right] = &\left[(\mathcal{L}'^{-1})^\dagger\nabla'_j\nabla'_i\mathbf{M_j}(\mathbf{x}')\nabla'_\mathbf{i} + (\mathcal{L}'^{-\mathbf{1}})^\dagger\nabla_\mathbf{j}\nabla_\mathbf{i}\mathbf{M_j}(\mathbf{x})\nabla_\mathbf{i}\right]\delta(x-x')\\ &- e\left(\frac{1}{m_i^3}-\frac{1}{m_e^3}\right)(\mathcal{L}'^{-1})^\dagger\nabla'\cdot[\mathbf{B}(\mathbf{x}')\times\nabla\delta(\mathbf{x}-\mathbf{x}')]\,,\end{aligned} \tag{6.66}$$

$$C_{11}^{-1}(\mathbf{x},\mathbf{x}') = 0\,. \tag{6.67}$$

It is easily verified that $\mathcal{L}^{-1}$ is the inverse of the operator $\mathcal{L}$ defined above. Although the inverse nabla operators invoked in the definition of C_{12}^{-1} and C_{21}^{-1} may not have an inherent mathematical/physical meaning, we leave them since their form might be useful. Once more, we point out that Eqs. (6.64)-(6.65) obey the symmetry relation of Eq. (6.33).

6.2.2.4 Towards the Dirac bracket

We note that the two elements necessary for computing the Dirac bracket are given below.

$$\{F, \Phi_1\} = \sum_s q_s \nabla_j \left(n_s F_{M_{s_j}} \right) , \tag{6.68}$$

$$\begin{aligned} \{F, \Phi_2\} = & - \sum_s \frac{q_s}{m_s} \left[\nabla_j \nabla_i \left(M_{s_i} F_{M_{s_j}} \right) + \nabla_j M_{s_i} \nabla_j F_{M_{s_i}} + M_{s_i} \Delta F_{M_{s_i}} \right] \\ & - \sum_s \frac{q_s}{m_s} \left[\nabla_i \rho_s \nabla_i F_{\rho_s} + \rho_s \Delta F_{\rho_s} \right] \\ & - \sum_s \frac{q_s}{m_s} \left[\nabla_i \sigma_s \nabla_i F_{\sigma_s} + \sigma_s \Delta F_{\sigma_s} \right] \\ & - \sum_s \frac{q_s^2}{m_s} \nabla_j \left(n_s F_{E_j} \right) + \sum_s \frac{q_s^2}{m_s} \nabla_j \left(n_s \epsilon_{jkl} B_k F_{M_{s_l}} \right) . \end{aligned} \tag{6.69}$$

We note that the antisymmetry of the bracket allows us to easily determine $\{\Phi_a, G\}$ where $a = 1, 2$. The functional derivatives of the Hamiltonian

are

$$\frac{\delta H}{\delta M_{s_j}} = \frac{M_{s_j}}{\rho_s}, \quad \frac{\delta H}{\delta E_j} = E_j, \quad \frac{\delta H}{\delta B_j} = B_j$$
$$\frac{\delta H}{\delta \rho_s} = -\frac{M_s^2}{2\rho_s^2} + U_s + \rho_s \frac{\partial U_s}{\partial \rho_s}, \quad \frac{\delta H}{\delta \sigma_s} = \rho_s \frac{\partial U_s}{\partial \sigma_s}. \tag{6.70}$$

Instead of directly writing down the bracket, it was our aim to compute some of the simpler dynamical equations and check whether the (numerous) calculations led us in the right direction.

On computing $\partial \mathbf{B}/\partial t$, we found that the Dirac bracket was the same as the Poisson bracket, i.e., we recovered Faraday's law and the additional terms in the Dirac bracket did not contribute; this was consistent with the expectations since Faraday's law must still be preserved.

Next, we computed $\{e(n_i - n_e), H\}_{DB}$, where the subscript DB denotes the Dirac bracket. After some manipulation, we find that it reduces to

$$\partial_t \left[e(n_i - n_e)\right] = \{e(n_i - n_e), H\}_{DB} = \{e(n_i - n_e), H\}_{PB} + \sum_s \frac{q_s}{m_s} \nabla \cdot \mathbf{M}_s \,,$$

and it must be noted that the very last term arises from the additional terms within the Dirac bracket. After simplifying further, we find that

$$\partial_t \left[e(n_i - n_e)\right] = \nabla \cdot \mathbf{J} - \nabla \cdot \mathbf{J} = 0, \tag{6.71}$$

and this does indeed verify that quasineutrality is preserved, if it is satisfied as an initial condition. Next, we determine the evolution equation for $\mathbf{E}$. This yields

$$\partial_t \mathbf{E} = \{\mathbf{E}, H\}_{DB} = \nabla \times \mathbf{B} - \left[\mathcal{I} - N(x) \nabla \mathcal{L}^{-1} \nabla \cdot\right] \mathbf{J}, \tag{6.72}$$

and $\mathcal{I}$ serves as the identity operator. Note that the above expression is reminiscent of a projection operator[15]. Let us take the divergence of the above term. The first one identically vanishes. The second also vanishes since we obtain $[\nabla \cdot \mathbf{J} - \mathcal{L}\mathcal{L}^{-1}\nabla \cdot \mathbf{J}]$, which also vanishes identically. As a result, we conclude that $\partial_t \nabla \cdot \mathbf{E} = 0$, which indicates that the electric field is always divergence free, if it is specified as an initial condition. This is in agreement with (6.71), which is to be expected as $e(n_i - n_e) - \nabla \cdot \mathbf{E}$ serves as a constraint. Therefore, the Dirac bracket satisfies all the imposed constraints, provided that the value of the two divergences, starts out as zero.

Our next aim was to compute the continuity equation for each species. We obtain:

$$\frac{\partial \rho_s}{\partial t} + \nabla_j M_{s_j} - \frac{1}{m_s^2}\nabla_j \left[\rho_s \nabla_j \left(\mathcal{L}^{-1}\left(\nabla_k M_{s_k}\right)\right)\right] = 0\,, \tag{6.73}$$

for each species s and $\mathcal{L}^{-1}$ was introduced in the matrix elements C_{ab} and their inverses. After some simplification, the above equation reduced to:

$$\frac{\partial \rho_s}{\partial t} + \nabla_j \left(\rho_s v_{s_j}\right) - \nabla_j \left(\frac{n_s^2 v_{s_j}}{N}\right) = 0\,, \tag{6.74}$$

where N is the operator defined in Eqs. (6.61)-(6.62) . From this, we compute the dynamical equation for $\delta n = n_i - n_e$, and find that

$$\frac{\partial\left(\delta n\right)}{\partial t} + \nabla_k \left[\left(\frac{m_e}{n_e} + \frac{m_i}{n_i}\right)^{-1}\left(m_i v_{i_k} - m_e v_{e_k}\right)\right] = 0\,, \tag{6.75}$$

and it is clear that the above equation is *not* the usual $\partial\left(\delta n\right)/\partial t + \nabla \cdot \mathbf{J} = 0$. However, what we *should* have obtained is $\partial\left(\delta n\right)/\partial t = 0$, and it is evident that

the above equation does *not* yield this condition except when $\mathbf{p}_e \equiv m_e \mathbf{v}_e = m_i \mathbf{v}_i \equiv \mathbf{p}_i$.

As we mentioned in the beginning of this chapter, the Dirac Constraints method, oftentimes results in equations of motion with generalized forces which restrict the system to evolve in a way that satisfies the constraints. At this moment, it is unclear to us whether this failure to obtain the usual continuity equation can be justified by those means. As a result, it remains an open problem to explain what exactly happens when one tries to enforce Eqs. (6.58)-(6.59) as constraints on the two-fluid bracket. In other words, is Eq. (6.74) an equation that can tell us something physical or not, and if not, why not?

Chapter 7

Conclusions and outline for future work

This work has been based on three distinct projects which illustrate the importance of Hamiltonian methods in plasma physics research and their efficacy for generating interesting and non-trivial results.

Chapters 1, 2 and 4 are introductory while, Chapters 3, 5 and 6 contain new results. In Chapter 1, we argued about the importance of having reduced systems that retain their Hamiltonian character and the benefits that the Hamiltonian formalism provides, once the Hamiltonian form of a system has been identified.

In Chapter 2, we gave a terse review of classical mechanics from the Hamiltonian perspective and provided a short introduction in the subject of noncanonical Hamiltonian systems.

In Chapter 3 we presented a Hamiltonian, five-field, electromagnetic gyrofluid model. We gave the Hamiltonian formulation of the model and found the families of Casimir invariants and the normal fields of it. We performed a linear study of the dispersion relation. We began by comparing the electrostatic dispersion relation with known analytic results for a slab ITG mode. We subsequently examined the electromagnetic properties of the model, including

toroidal curvature, by comparing the dependence of the growth rate on well known stabilizing factors and comparing them with the local kinetic result of Ref. [62]. We found good qualitative agreement.

In Chapter 4, we gave a brief reminder of the action principle of classical mechanics and it's use in fluid and plasma models. We also introduced and explained the Lagrangian and Eulerean frameworks of fluid dynamics.

In Chapter 5, we derived several fluid models from a general two-fluid action functional. All approximations, ordering schemes, and changes of variables were done in the action functional before Hamilton's principle was invoked. We defined a new set of Lagrangian variables, and under the assumption of quasineutrality, we constructed a new set of nonlocal Lagrange-Euler maps assuring that our Lagrangian equations of motion can be Eulerianized. Lastly, we derived several conservation laws for these models using Noether's theorem.

In Chapter 6, we presented two worked examples of the method of Dirac for imposing constraints on Hamiltonian systems while preserving the Hamiltonian structure. The first example was on an electrostatic model that has Hasegawa-Mima and RMHD as distinct limits. We imposed the constraint of electron adiabaticity along with a constraint on the form of the poloidal flux and displayed the reduced system. The second example we showed was the imposition of the constraints of quasineutrality and charge conservation on the two-fluid bracket. After following the steps of the method, although we confirmed that the Dirac bracket preserves the constraints, while calculating the

continuity equation, we stumbled upon an unphysical result. This outcome, brought into question the validity of our approach.

The three main results of the thesis, give rise to new questions and open up possibilities for further inquiry.

First, the five-field model presented in Chapter 3 can be extended to include electron temperature. This straightforward improvement would make possible a direct comparison with Ref. [62] and further enhance it's quantitative accuracy. Moreover, the fact that the value of critical η for the onset of instability depends on the adiabatic index, strongly suggests that the inclusion of an evolution equation for $p_\perp$, in a manner that preserves the Hamiltonian character, would endow the model with the correct asymptotic behavior in the low $k_\perp$ limit. Also, from the perspective of Hamiltonian theory, a weakly nonlinear system can be extracted from the five-field model, following the method presented in Ref. [95].

In addition, the combination of the agility and speed of gyrofluid models with the guarantee of conservation laws that Hamiltonian systems offer, makes the five-field model an ideal tool for the study of multi-scale, nonlinear phenomena where correct accounting of energy transfers, past the linear stage, is crucial. Uncertainty quantification (UQ) provides another application that is worth mentioning. The large number of inputs to gyrokinetic codes make comprehensive UQ impractical, but the existence of a reduced model opens up new avenues for charting model sensitivities and subsequently using Bayesian inference with a smaller number of runs of the GK code to selectively refine

the predictions and reduce the error bars. Therefore, it would be reasonable to implement the five-field model (or future improvements of it) within an existing framework, like BOUT++ or to develop a solver for it.

Moving on to the material of Chapter 5, we acknowledge that Extended MHD has already been the subject of intense activity: it's bracket structure has been identified in Ref. [1] and in Ref. [21] from an entirely Lagrangian standpoint. Casting Extended MHD in Hamiltonian formalism has facilitated the study of it's topological structures (see e.g. Ref. [70] for Extended MHD and Ref. [7] for Hall MHD) and has elucidated connections between various reduced MHD models [72].

Of course, the procedure of systematic orderings applied directly in the action can be utilized to produce new reduced models. Examples where the method has led to the formulation of hitherto unknown models (or at least, unknown Hamiltonian versions of known models) include single fluid models with gyroviscocity [92, 71] and the identification of a variant of Extended MHD, called inertial MHD [73].

The novel nonlocal Lagrange-Euler map of Chapter 5 is of particular general importance. Usual Lagrange-Euler maps (also known as momentum maps) entail the advection of various quantities by a single velocity field and this can be traced to the algebraic structure of the Poisson bracket written in terms of Eulerian variables (see e.g. Ref. [83]). For single fluid models like MHD the Poisson bracket[87] has semi-direct product structure, which occurs in a variety of fluid contexts (e.g. Refs. [108, 51, 78]). However, many systems

do not possess this semi-direct product structure (e.g. Refs. [40, 31, 124, 94]) and indeed a general theory of algebraic extensions was given in Ref. [126]. It is the selection of the set of observables and the Eulerian Closure Principle that give rise to the general algebras underlying Poisson brackets. Detailed construction of general algebras of Ref. [126] is a topic of ongoing research.

Finally, the unusual results of Chapter 6, raise the issue of whether the method of Dirac Constraints is the proper way of addressing the imposition of quasineutrality and charge conservation as constraints in the two-fluid bracket. The uncommon form of the obtained continuity equation may be the result of unphysical generalized forces introduced by the procedure of constraining the system. Undoubtedly, it might also be the case that there exists some "correct" way of applying the method that circumvents the problem of unphysical generalized forces. If such a way exists, it is unknown to us and as it has often been the case with similar issues, the point might prove highly non-trivial and fruitful.

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